

HOLOMORPHIC CURVES IN EXPLODED MANIFOLDS - KURANISHI STRUCTURE

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ABSTRACT. This paper constructs a concrete kind of Kuranishi structure for the moduli stack of holomorphic curves in an exploded manifold. This Kuranishi structure is embedded inside a moduli stack of $C^{\infty,1}$ curves in order to avoid technicalities which arise when working with abstract Kuranishi structures. The construction also works for the moduli stack of holomorphic curves in any compact symplectic manifold.

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1. INTRODUCTION

In this paper, we shall construct a concrete kind of Kuranishi structure on the moduli stack of holomorphic curves in exploded manifolds. Because the category of exploded manifolds extends the category of smooth manifolds, this construction includes a new construction of Kuranishi structures on the moduli stack of holomorphic curves in compact symplectic manifolds. Definitions for exploded manifolds can be found in [14], and compactness results for the moduli stack of holomorphic curves in exploded manifolds are found in [11].

Fukaya and Ono used Kuranishi structures in their definition of Gromov-Witten invariants in [3]. Our Kuranishi structures will differ from Fukaya and Ono's in that they are embedded into a moduli stack of $C^{\infty,1}$ curves. $C^{\infty,1}$ indicates a kind of regularity, which for all practical purposes, the unfamiliar reader may regard as meaning 'smooth'. The reader who is unfamiliar with stacks, but who likes to imagine a Fréchet orbifold or polyfold structure on a space of maps may think of our Kuranishi structures as being embedded in such a space.

Our embedded Kuranishi structures avoid some technicalities involved in using Kuranishi structures to define Gromov-Witten invariants, and are conceptually easier to deal with than the Kuranishi structures defined in [3], which can be a little slippery. See the preprint [10] by McDuff and Werheim for some discussion of issues that must be overcome when using abstract Kuranishi structures, and see the recent preprint [2] of Fukaya, Oh, Ohta, and Ono for some improvements on Fukaya and Ono's original definitions and a much more detailed version of their construction of Gromov-Witten invariants.

In Fukaya and Ono's original paper, [3], they use a homotopy of the linearization $D\bar{\partial}$ of the $\bar{\partial}$ operator to a complex map in order to orient their Kuranishi structures, and also to construct a stably almost complex structure. In [4], Fukaya and Ono sketched how, if their stably almost complex structure was globally defined, then they could define invariant integer counts of holomorphic curves. In the preprint [7], Dominic Joyce sketched a related construction of integer invariants using similar almost complex information with a modified version of Kuranishi structures. In the category of smooth manifolds, the construction of a stably almost complex structure must be treated with great care around singular curves. In the category of exploded manifolds, no curves are singular in this way, and the argument constructing a globally defined stably almost complex structure using a homotopy of $D\bar{\partial}$ to a complex map is relatively straightforward.

Very roughly, the idea of an embedded Kuranishi structure is as follows: The $\bar{\partial}$ equation may be regarded as giving a section of a sheaf \mathcal{Y} over a moduli stack, \mathcal{M}^{st} , of stable $C^{\infty,1}$ curves. In an open neighborhood \mathcal{O} around a holomorphic curve f , a nice subsheaf $V \subset \mathcal{Y}$ may be chosen, which may be thought of as a finite dimensional vector bundle over \mathcal{O} . If $D\bar{\partial}$ is transverse to V at f , then in an open neighborhood \mathcal{U} of f , the moduli stack of solutions h to $\bar{\partial}h \in V$ may be represented by the quotient of some $C^{\infty,1}$ family of curves \hat{f} by a group G of automorphisms. Our embedded Kuranishi charts have the information $(\mathcal{U}, V, \hat{f}/G)$.

By putting further assumptions on V , we may construct embedded Kuranishi charts which have nice properties such as being compatible with chosen evaluation maps, or having an equivalent of Fukaya and Ono's stably almost complex structure.

For two embedded Kuranishi $(\mathcal{U}_i, V_i, \hat{f}_i/G_i)$ charts to be compatible, we require that on the intersection of \mathcal{U}_1 with \mathcal{U}_2 , either V_1 must be a subsheaf of V_2 or V_2

must be a subsheaf of V_1 . There are obviously much weaker versions of compatibility which would suffice for constructing Gromov-Witten invariants, but subject to compactness assumptions on the moduli stack of holomorphic curves, a covering by such compatible embedded Kuranishi charts always exists, and any two such embedded Kuranishi structures are homotopic, so it is not necessary to use a weaker version of compatibility.

To construct objects on a Kuranishi structure such as the weighted branched perturbation of $\bar{\partial}$ used by Fukaya and Ono to define Gromov-Witten invariants, a procedure of transfinite induction can be used. The idea is to choose the perturbation in order, starting with the Kuranishi charts $(\mathcal{U}, V, \hat{f}/G)$ for which V has minimal dimension. In order to be able to define a perturbation on a Kuranishi chart which is compatible with all the previously chosen perturbations, we require all our Kuranishi charts to have compatible extensions, and define our perturbations on extensions in order to control the behavior of perturbations near the boundary of Kuranishi charts.

The structure of this paper is as follows: In section 2, we go over some basics about the moduli stack $\mathcal{M}^{\infty,1}$ of $C^{\infty,1}$ curves in exploded manifolds, and an open substack \mathcal{M}^{st} of well behaved stable curves. Reading section 2 is essential for understanding what is proved in this paper. The meaning of maps from $\mathcal{M}^{\infty,1}$ to exploded manifolds or orbifolds is spelled out in section 2.3. A very natural topology on $\mathcal{M}^{\infty,1}$ is defined in section 2.4, and identified with the slightly strange seeming topology used on $\mathcal{M}^{\infty,1}$ in [11]. The tangent space of \mathcal{M}^{st} is defined and demystified in section 2.5, then the linearization $D\bar{\partial}$ of the $\bar{\partial}$ operator on \mathcal{M}^{st} is defined at holomorphic curves in section 2.6. Embedded Kuranishi structures are defined in section 2.7.

Section 3 contains a quick summary of the results of [13] which shall be necessary for this paper. Sections 4.1 and 4.2 construct some evaluation maps from \mathcal{M}^{st} . Section 5 defines an important notion of a core family, which gives a concrete local model for \mathcal{M}^{st} , which is then used continuously throughout the rest of the paper. Section 5.2 proves a technical lemma about the topology of \mathcal{M}^{st} which is essential for showing that Kuranishi charts may be shrunk in order to avoid degenerate behavior at their boundaries. Section 6 is dedicated to locally analyzing the moduli stack of solutions to the weakened $\bar{\partial}$ equation, $\bar{\partial}f \in V$.

Section 7 constructs an embedded Kuranishi structure for the moduli stack of holomorphic curves in any family of exploded manifolds $\hat{\mathbf{B}} \rightarrow \mathbf{B}_0$ satisfying the compactness condition that the map to \mathbf{B}_0 from the moduli stack of holomorphic curves in any connected component of \mathcal{M}^{st} is proper. It is also proved that any two such Kuranishi structures are homotopic. Section 8 constructs a version of Fukaya and Ono's stably almost complex structure for our embedded Kuranishi structures.

2. STRUCTURE OF THE MODULI STACK OF STABLE CURVES

In this section, we shall work towards describing what an embedded Kuranishi structure is by describing basic properties of the moduli stack $\mathcal{M}^{\infty,1}$ of $C^{\infty,1}$ curves, concentrating mostly on the well behaved open substack $\mathcal{M}^{st} \subset \mathcal{M}^{\infty,1}$ of stable $C^{\infty,1}$ curves. We shall use the notation $\mathcal{M} \subset \mathcal{M}^{st}$ for the moduli stack of stable holomorphic curves.

2.1. The functors \mathbf{F} and \mathbf{C} .

This paper studies families of holomorphic curves in a smooth family of targets in the exploded category,

$$\pi_{\mathbf{B}_0} : (\hat{\mathbf{B}}, J, \omega) \longrightarrow \mathbf{B}_0$$

where each fiber of $\pi_{\mathbf{B}_0}$ is a complete, basic exploded manifold with a $\bar{\partial}$ log-compatible complex structure J tamed by a symplectic form ω (using terminology from [14] and [11]). We will often talk about $C^{\infty,1}$ families of curves \hat{f} in $\hat{\mathbf{B}} \longrightarrow \mathbf{B}_0$ which will correspond to commutative diagrams

$$\begin{array}{ccc} \mathbf{C}(\hat{f}) & \xrightarrow{\hat{f}} & \hat{\mathbf{B}} \\ \downarrow \pi_{\mathbf{F}(\hat{f})} & & \downarrow \pi_{\mathbf{B}_0} \\ \mathbf{F}(\hat{f}) & \longrightarrow & \mathbf{B}_0 \end{array}$$

where \hat{f} , and all other maps in the above diagram are $C^{\infty,1}$ maps, and $\pi_{\mathbf{F}(\hat{f})} : \mathbf{C}(\hat{f}) \longrightarrow \mathbf{F}(\hat{f})$ is a family of curves with a fiberwise complex structure j (as defined in [14]).

We shall think of \mathbf{F} and \mathbf{C} as functors in the following way: As noted in section 11 of [14], families of $C^{\infty,1}$ curves in $\hat{\mathbf{B}} \longrightarrow \mathbf{B}_0$ form a category with morphisms $\hat{f} \longrightarrow \hat{g}$ given by commutative diagrams

$$\begin{array}{ccccc} & & \hat{f} & & \\ & \nearrow & & \searrow & \\ \mathbf{C}(\hat{f}) & \longrightarrow & \mathbf{C}(\hat{g}) & \xrightarrow{\hat{g}} & \hat{\mathbf{B}} \\ \downarrow \pi_{\mathbf{F}(\hat{f})} & & \downarrow \pi_{\mathbf{F}(\hat{g})} & & \\ \mathbf{F}(\hat{f}) & \longrightarrow & \mathbf{F}(\hat{g}) & & \end{array}$$

so that restricted to each fiber of $\pi_{\mathbf{F}(\hat{f})}$ and $\pi_{\mathbf{F}(\hat{g})}$, the map $\mathbf{C}(\hat{f}) \longrightarrow \mathbf{C}(\hat{g})$ is a holomorphic isomorphism. In this way, \mathbf{C} and \mathbf{F} can be regarded as functors from the category of $C^{\infty,1}$ families of curves to the category of $C^{\infty,1}$ exploded manifolds.

Use the notation $\mathcal{M}^{\infty,1}(\hat{\mathbf{B}})$ for the moduli stack of $C^{\infty,1}$ curves in $\hat{\mathbf{B}}$, which is the category of $C^{\infty,1}$ families of curves together with the functor \mathbf{F} . When no ambiguity is present, simply use the notation $\mathcal{M}^{\infty,1}$.

Throughout this paper, a substack $\mathcal{U} \subset \mathcal{M}^{\infty,1}(\hat{\mathbf{B}})$ shall always mean a full subcategory of $\mathcal{M}^{\infty,1}(\hat{\mathbf{B}})$ so that a family of curves \hat{f} in $\mathcal{M}^{\infty,1}(\hat{\mathbf{B}})$ is in \mathcal{U} if and only if each individual curve in \hat{f} is in \mathcal{U} .

2.2. The sheaf \mathcal{Y} over the moduli stack of curves.

We shall be considering holomorphic curves in $\mathcal{M}^{\infty,1}(\hat{\mathbf{B}})$ as the solution of an equation $\bar{\partial}\hat{f} = 0$. In this section, we shall consider what should be the range of $\bar{\partial}$ on the moduli stack of $C^{\infty,1}$ curves. This shall be a sheaf \mathcal{Y} over the moduli stack of curves.

We shall often need to consider the vertical (co)tangent space of families or submersions.

Definition 2.1. *Given a submersion*

$$f : \mathbf{A} \longrightarrow \mathbf{B}$$

use the notation $T\mathbf{A}\downarrow_{\mathbf{B}}$ or $T_{\text{vert}}\mathbf{A}$ to indicate the vertical tangent bundle of \mathbf{A} over \mathbf{B} , which is the sub bundle of $T\mathbf{A}$ consisting of the kernel of df .

Use the notation $T^\mathbf{A}\downarrow_{\mathbf{B}}$ or $T_{\text{vert}}^*\mathbf{A}$ to indicate the vertical cotangent bundle of \mathbf{A} over \mathbf{B} , which equal to the dual of $T\mathbf{A}\downarrow_{\mathbf{B}}$.*

We shall use the notation T_{vert} when no ambiguity shall arise, and when it is less cumbersome. For example, the notation $T_{vert}\mathbf{C}(\hat{f})$ shall always mean $T\mathbf{C}(\hat{f})|_{\mathbf{F}(\hat{f})}$, and given a family of targets, $\hat{\mathbf{B}} \rightarrow \mathbf{B}_0$, the notation $T_{vert}\hat{\mathbf{B}}$ shall always mean $T\hat{\mathbf{B}}|_{\mathbf{B}_0}$.

Definition 2.2. Given a $C^{\infty,1}$ family of curves \hat{f} in $\hat{\mathbf{B}}$, define

$$d_{vert}\hat{f} : T_{vert}\mathbf{C}(\hat{f}) \rightarrow T_{vert}\hat{\mathbf{B}}$$

to be $d\hat{f}$ restricted to the vertical tangent space, $T_{vert}\mathbf{C}(\hat{f}) \subset T\mathbf{C}(\hat{f})$.

Define

$$\bar{\partial}\hat{f} : T_{vert}\mathbf{C}(\hat{f}) \rightarrow T_{vert}\hat{\mathbf{B}}$$

as

$$\bar{\partial}\hat{f} := \frac{1}{2}(d_{vert}\hat{f} + J \circ d_{vert}\hat{f} \circ j)$$

We shall consider $\bar{\partial}\hat{f}$ as a section of the vector bundle $(T_{vert}^*\mathbf{C}(\hat{f}) \otimes \hat{f}^*T_{vert}\hat{\mathbf{B}})^{0,1}$ over $\mathbf{C}(\hat{f})$, and also as a section of a corresponding sheaf over $\mathbf{F}(\hat{f})$. As the bundle $(T_{vert}^*\mathbf{C}(\hat{f}) \otimes \hat{f}^*T_{vert}\hat{\mathbf{B}})^{0,1}$ is cumbersome to write out in full, use the following notation:

Definition 2.3. Use the notation

$$Y(\hat{f}) := (T_{vert}^*\mathbf{C}(\hat{f}) \otimes \hat{f}^*T_{vert}\hat{\mathbf{B}})^{0,1}$$

which is the sub vector bundle of $T_{vert}^*\mathbf{C}(\hat{f}) \otimes_{\mathbb{R}} \hat{f}^*T_{vert}\hat{\mathbf{B}}$ consisting of vectors so that the action of J on the second factor is equal to -1 times the action of j on the first factor.

As well as the vector bundle, $Y(\hat{f})$, we shall consider the sheaf

$$\mathcal{Y}(\hat{f})$$

which is a sheaf of $C^{\infty,1}(\mathbf{F}(\hat{f}))$ -modules over $\mathbf{F}(\hat{f})$ consisting of $C^{\infty,1}$ sections of $Y(\hat{f})$ which vanish on all integral vectors within $T_{vert}\mathbf{C}(\hat{f})$.

Because $\bar{\partial}\hat{f}$ always vanishes on integral vectors in $T_{vert}\mathbf{C}(\hat{f})$, $\bar{\partial}\hat{f}$ is a global section of $\mathcal{Y}(\hat{f})$.

Note that given any map

$$\hat{f} \rightarrow \hat{g}$$

there is a corresponding pullback diagram of vector bundles

$$\begin{array}{ccc} Y(\hat{f}) & \longrightarrow & Y(\hat{g}) \\ \downarrow & & \downarrow \\ \mathbf{C}(\hat{f}) & \longrightarrow & \mathbf{C}(\hat{g}) \end{array}$$

and a functorial map of sheaves

$$\mathcal{Y}(\hat{f}) \leftarrow \mathcal{Y}(\hat{g})$$

We shall regard \mathcal{Y} as a sheaf over $\mathcal{M}^{\infty,1}$.

2.3. Maps from the moduli stack of curves.

As spelled out in section 11 of [14], we may consider any exploded manifold \mathbf{A} as a stack $\mathcal{S}(\mathbf{A})$ over the category of $C^{\infty,1}$ exploded manifolds with objects consisting of $C^{\infty,1}$ maps of exploded manifolds into \mathbf{A} . The correct notion of a map $\mathcal{M}^{\infty,1} \rightarrow \mathbf{A}$ is a map of stacks over the category of $C^{\infty,1}$ exploded manifolds

$$\Phi : \mathcal{M}^{\infty,1} \rightarrow \mathcal{S}(\mathbf{A})$$

In particular, such a map associates to every family \hat{f} of curves in $\mathcal{M}^{\infty,1}$ a $C^{\infty,1}$ map

$$\mathbf{F}(\hat{f}) \xrightarrow{\Phi_{\hat{f}}} \mathbf{A}$$

so that given any map $\hat{g} \rightarrow \hat{f}$, the following diagram commutes

$$\begin{array}{ccc} & \Phi_{\hat{g}} & \\ \mathbf{F}(\hat{g}) & \xrightarrow{\quad} & \mathbf{F}(\hat{f}) \xrightarrow{\Phi_{\hat{f}}} \mathbf{A} \end{array}$$

Similarly, a continuous map from $\mathcal{M}^{\infty,1}$ to a topological space X is for every family \hat{f} of curves, a continuous map

$$\mathbf{F}(\hat{f}) \rightarrow X$$

so that given any map $\hat{g} \rightarrow \hat{f}$, the following diagram commutes

$$\begin{array}{ccc} & & \\ \mathbf{F}(\hat{g}) & \xrightarrow{\quad} & \mathbf{F}(\hat{f}) \longrightarrow X \end{array}$$

Lemma 2.8 on page 9 states that an open subset \mathcal{U} of $\mathcal{M}^{\infty,1}$ using the topology on $\mathcal{M}^{\infty,1}$ defined in [14] corresponds to an open subset $U(\hat{f})$ of $\mathbf{F}(\hat{f})$ for all families \hat{f} in $\mathcal{M}^{\infty,1}$ so that every map $\hat{g} \rightarrow \hat{f}$ pulls back $U(\hat{f})$ to $U(\hat{g})$. This implies that the above definition of a continuous map is equivalent to a continuous map from the topological space $\mathcal{M}^{\infty,1}$ to X when $\mathcal{M}^{\infty,1}$ is given the structure of a topological space defined in [14].

We also want to consider maps of $\mathcal{M}^{\infty,1}$ to finite dimensional exploded orbifolds. We shall always think of such orbifolds as Deligne-Mumford stacks in the category of exploded manifolds. Such stacks are locally equivalent to the following example:

If a finite group G acts on \mathbf{A} , then $\mathcal{S}(\mathbf{A}/G)$ is a stack with objects consisting of G -equivariant $C^{\infty,1}$ maps of G -bundles into \mathbf{A} and the correct notion of a map of $\mathcal{M}^{\infty,1}$ into \mathbf{A}/G is a map $\mathcal{M}^{\infty,1} \rightarrow \mathcal{S}(\mathbf{A}/G)$. In particular, such a map associates to every family \hat{f} of curves in $\mathcal{M}^{\infty,1}$ a G -bundle

$$\hat{f}' \rightarrow \hat{f}$$

and a G -equivariant, $C^{\infty,1}$ map

$$\mathbf{F}(\hat{f}') \rightarrow \mathbf{A}$$

and associates to any map of curves $\hat{g} \rightarrow \hat{f}$, a G -equivariant map $\hat{g}' \rightarrow \hat{f}'$ so that the following G -equivariant diagram commutes

$$\begin{array}{ccccc} & & & & \\ & & \mathbf{F}(\hat{f}') & \longrightarrow & \mathbf{A} \\ & \mathbf{F}(\hat{g}') & \longrightarrow & \mathbf{F}(\hat{f}') & \longrightarrow \mathbf{A} \\ \downarrow & & & \downarrow & \\ \mathbf{F}(\hat{g}) & \longrightarrow & \mathbf{F}(\hat{f}) & & \end{array}$$

and so that given a composition $\hat{h} \longrightarrow \hat{g} \longrightarrow \hat{f}$, the G -equivariant diagram

$$\begin{array}{ccccc} \hat{h}' & \xrightarrow{\quad} & \hat{g}' & \xrightarrow{\quad} & \hat{f}' \\ & \searrow & & \nearrow & \\ & & & & \end{array}$$

commutes.

Remark 2.4. *As explained in [9] and [6], a good way of thinking of smooth orbifolds is as Deligne-Mumford stacks over the category of smooth manifolds. The reader who finds the definitions of stacks scary, should just think of this as defining orbifolds by saying what maps of manifolds into them are. (For example, in the case of Deligne-Mumford space, a map of a manifold M into Deligne-Mumford space is a family of stable curves over M , so it is completely natural to think of Deligne-Mumford space as a stack in this way.)*

In our case, we shall define (smooth or $C^{\infty,1}$) exploded orbifolds as stacks over the category of (smooth or $C^{\infty,1}$) exploded manifolds which are Deligne-Mumford stacks in the sense that they are locally equivalent to $\mathcal{S}(\mathbf{A}/G)$ for some exploded manifold \mathbf{A} with an action of the finite group G .

In section 4.1, we show that $\mathcal{M}(pt)$ is an exploded orbifold equal to the explosion of Deligne-Mumford space, and construct a map $\mathcal{M}^{st} \longrightarrow \mathcal{M}(pt)$.

The maps from \mathcal{M}^{st} we have defined so far all involve maps from $\mathbf{F}(\hat{f})$ for each individual family \hat{f} . We shall also need a notion of a map from \mathcal{M}^{st} which involves maps from $\mathbf{C}(\hat{f})$ for each individual family \hat{f} .

Given any stack of curves $\mathcal{U} \subset \mathcal{M}^{\infty,1}$, we shall use the notation $\mathcal{U}^{+1} \longrightarrow \mathcal{U}$ for the universal curve over \mathcal{U} , where \mathcal{U}^{+1} should be regarded as the stack of curves obtained from curves in \mathcal{U} by adding an extra puncture. (See section 4.2 for the construction of the lift \hat{f}^{+1} of a family of curves in \mathcal{U} to a family in \mathcal{U}^{+1} with an extra marked point.)

Definition 2.5. *Let \mathcal{U} be a substack of $\mathcal{M}^{\infty,1}(\hat{\mathbf{B}})$, and let $\hat{\mathbf{A}} \longrightarrow \mathbf{X}$ be a family of almost complex exploded manifolds with a choice of a finite group G of automorphisms. Define a fiberwise holomorphic map with effective G action*

$$\begin{array}{ccc} \mathcal{U}^{+1} & \longrightarrow & \hat{\mathbf{A}}/G \\ \downarrow & & \downarrow \\ \mathcal{U} & \longrightarrow & \mathbf{X}/G \end{array}$$

to be for every \hat{f} in \mathcal{U} a choice of G -fold cover \hat{f}' of \hat{f} and a G -equivariant family of holomorphic curves

$$\begin{array}{ccc} \mathbf{C}(\hat{f}') & \longrightarrow & \hat{\mathbf{A}} \\ \downarrow & & \downarrow \\ \mathbf{F}(\hat{f}') & \longrightarrow & \mathbf{X} \end{array}$$

so that for any map $\hat{g} \longrightarrow \hat{f}$, the following are G -equivariant pullback diagrams:

$$\begin{array}{ccc} \hat{g}' & \longrightarrow & \hat{f}' \\ \downarrow & & \downarrow \\ \hat{g} & \longrightarrow & \hat{f} \end{array}$$

$$\begin{array}{ccccc} \mathbf{C}(\hat{g}') & \longrightarrow & \mathbf{C}(\hat{f}') & \longrightarrow & \hat{\mathbf{A}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{F}(\hat{g}') & \longrightarrow & \mathbf{F}(\hat{f}') & \longrightarrow & \mathbf{X} \end{array}$$

and so that the action of G is effective in the sense that the map $\mathbf{C}(f') \rightarrow \mathbf{A}$ is never preserved by the action of a nontrivial element of G acting on $\hat{\mathbf{A}}$.

Remark 2.6. Without the assumption of an effective G action, we would need to be more careful about specifying that $\hat{g}' \rightarrow \hat{f}'$ is determined in a functorial way. With the assumption of an effective G action, there is only one map $\hat{g}' \rightarrow \hat{f}'$ so that the required diagrams commute.

Section 4.1 constructs a fiberwise holomorphic map

$$\begin{array}{ccc} \mathcal{M}^{\infty,1}(\hat{\mathbf{B}})^{+1} & \longrightarrow & \mathcal{M}(pt)^{+1} \\ \downarrow & & \downarrow \\ \mathcal{M}^{\infty,1}(\hat{\mathbf{B}}) & \longrightarrow & \mathcal{M}(pt) \end{array}$$

This map may be insufficient for our purposes because it collapses bubbles in the domain of curves. In section 5, we shall construct ‘core families’ \hat{f} with a group of automorphisms G so that there is a neighborhood \mathcal{U} of \hat{f} with a fiberwise holomorphic map

$$\begin{array}{ccc} \mathcal{U}^{+1} & \longrightarrow & \mathbf{C}(\hat{f})/G \\ \downarrow & & \downarrow \\ \mathcal{U} & \longrightarrow & \mathbf{F}(\check{f})/G \end{array}$$

so that for any curve f in U , the corresponding map $|G|$ maps $\mathbf{C}(f) \rightarrow \mathbf{C}(\hat{f})$ are holomorphic isomorphisms onto curves in $\mathbf{C}(\hat{f})$.

2.4. Topology of the moduli stack of curves.

A family of curves \hat{f} comes with a naturally defined topology on $\mathbf{F}(\hat{f})$. The topology on $\mathcal{M}^{\infty,1}$ was described in [14] in terms of convergence of a sequence of curves. In this section, we show that open substacks \mathcal{U} of $\mathcal{M}^{\infty,1}$ correspond to substacks which intersect any family of curves in an open subset.

Lemma 2.7. *Given any sequence of curves in $\mathcal{M}^{\infty,1}$ converging to f in $C^{\infty,1}$, there exists a family \hat{f} of curves containing f and a subsequence f_i of the given sequence of curves so that within $\mathbf{F}(\hat{f})$, f_i converges to f .*

Proof:

The definition of the $C^{\infty,1}$ topology on $\mathcal{M}^{\infty,1}$ from sections 7 and 11 of [14] implies that there exists

- a family of curves \hat{g} containing f ,
- a sequence of curves f'_i in \hat{g} converging to f ,
- a sequence of fiberwise almost complex structures j_i on $\mathbf{C}(\hat{g})$ converging in $C^{\infty,1}$ to the given almost complex structure on $\mathbf{C}(\hat{g})$,
- and a sequence of sections ψ_i of $\hat{g}^*T_{vert}\hat{\mathbf{B}}$ converging in $C^{\infty,1}$ to 0

so that there is an identification of $\mathbf{C}(f_i)$ with $\mathbf{C}(f'_i)$ with the almost complex structure j' , and so that the map f_i is f'_i followed by exponentiation of ψ_i in some metric.

Let \hat{f}_0 be $\hat{g} \times \mathbb{R}$. The section ψ_i being small in $C^{\infty,1}$ implies that there exists a section ψ'_i of $\hat{f}_0^* T_{vert} \hat{\mathbf{B}} \subset \hat{g} \times \mathbb{R}$ which is equal to ψ at $\hat{g} \times \{i^{-1}\}$, supported within the region $\hat{g} \times ((i+1)^{-1}, (i-1)^{-1})$, and small in $C^{\infty,1}$. As noted in [14] and [13], convergence in $C^{\infty,1}$ for fiberwise complex structures on $\mathbf{C}(\hat{f}_0)$ or sections of $\hat{f}_0^* T_{vert} \hat{\mathbf{B}}$ is equivalent to convergence in some countable sequence of norms. By passing to a subsequence, we may assume that this ψ'_i has size less than 2^{-i} in the first i norms. Then $\sum_i \psi'_i$ is a $C^{\infty,1}$ section of $\hat{f}_0^* T_{vert} \hat{\mathbf{B}}$, which restricts to be ψ_i on $\hat{g} \times \{i^{-1}\}$ and which is zero on $\hat{g} \times \{0\}$. Define the map \hat{f} to be \hat{f}_0 followed by exponentiation of $\sum_i \psi'_i$.

Similarly, by passing to a subsequence we may construct a $C^{\infty,1}$, fiberwise complex structure j on $\mathbf{C}(\hat{f})$ which restricts to $\mathbf{C}(\hat{g}) \times \{i^{-1}\}$ to be j_i , and which is the original fiberwise complex structure on $\mathbf{C}(\hat{g}) \times \{0\}$.

Now there is a sequence of inclusions $f_i \rightarrow \hat{f}$ converging to $f \rightarrow \hat{f}$, as required. \square

Lemma 2.8. *A substack \mathcal{U} of $\mathcal{M}^{\infty,1}$ is open if and only if for all families \hat{f} in $\mathcal{M}^{\infty,1}$, the subset of $\mathbf{F}(\hat{f})$ consisting of curves in \mathcal{U} is open.*

Proof:

Suppose that \mathcal{U} is a substack of $\mathcal{M}^{\infty,1}$. Denote by $U(\hat{f})$ the subset of $\mathbf{F}(\hat{f})$ consisting of curves in \mathcal{U} .

Suppose that \mathcal{U} is open. Then convergence within \hat{f} is at least as strong as convergence within $\mathcal{M}^{\infty,1}$, so $U(\hat{f}) \subset \mathbf{F}(\hat{f})$ is open.

Suppose that $U(\hat{f}) \subset \mathbf{F}(\hat{f})$ is open for all families \hat{f} in $\mathcal{M}^{\infty,1}$. Then if f_i is a sequence of curves converging to f , Lemma 2.7 implies that there exists a subsequence converging to f within some family \hat{f} , so that subsequence is eventually contained in $U(\hat{f})$, and therefore eventually contained in \mathcal{U} . This implies that every sequence of curves converging to a curve in \mathcal{U} is eventually contained in \mathcal{U} , so \mathcal{U} is open. \square

2.5. Tangent space of \mathcal{M}^{st} .

In this section, we shall define the tangent space of the open substack $\mathcal{M}^{st} \subset \mathcal{M}^{\infty,1}$ of stable curves. We shall first discuss the tangent space to a single curve f in a single target \mathbf{B} , and then define the relative tangent space in the case of a family of targets $\hat{\mathbf{B}} \rightarrow \mathbf{B}_0$ and the (relative) tangent sheaf in the case of a family of curves.

2.5.1. Stability.

Definition 2.9. *Call a $C^{\infty,1}$ curve $f : \mathbf{C}(f) \rightarrow \hat{\mathbf{B}}$ stable if it has only a finite number of automorphisms and its smooth part $[f]$ has only a finite number of automorphisms.*

Let \mathcal{M}^{st} be the substack of $\mathcal{M}^{\infty,1}$ consisting of families of stable curves.

Remark 2.10. *Note that the condition that f has a finite number of automorphisms is only necessary if $\mathbf{C}(f) = \mathbf{T}$, as otherwise every nontrivial automorphism of f is also a nontrivial automorphism of $[f]$. There are a number of possible candidates for a definition of a stable curve f . A weaker definition which agrees with the above on holomorphic curves is to require that f has only a finite number of automorphisms and that f is not a nontrivial refinement of another curve. A much weaker definition is to just require that f has a finite number of automorphisms. The stack of curves satisfying this much weaker definition of stability is not sufficiently well behaved.*

For example, the stack of ‘stable’ curves mapping to a point is an exploded orbifold using the two stronger definitions, but not the much weaker definition of ‘stable’.

2.5.2. $T_f\mathcal{M}^{st}(\mathbf{B})$.

Given a curve $f : \mathbf{C} \rightarrow \mathbf{B}$, the tangent space $T_f\mathcal{M}^{st}$ of \mathcal{M}^{st} at f shall be defined using a short exact sequence

$$0 \rightarrow \Gamma(T\mathbf{C}) \rightarrow \Gamma^{0,1}(T\mathbf{C} \otimes T^*\mathbf{C}) \times \Gamma(f^*T\mathbf{B}) \rightarrow T_f\mathcal{M}^{st} \rightarrow 0$$

In what follows, we shall discuss the terms above.

Let $\Gamma(f^*T\mathbf{B})$ denote the space of $C^{\infty,1}$ sections of $f^*T\mathbf{B}$. The action of J on sections of $f^*T\mathbf{B}$ makes $\Gamma(f^*T\mathbf{B})$ into a complex vector space. $\Gamma(f^*T\mathbf{B})$ can be regarded as the infinitesimal variations of a map from a fixed domain \mathbf{C} .

Let $\Gamma^{0,1}(T\mathbf{C} \otimes T^*\mathbf{C})$ denote the space of $C^{\infty,1}$ sections α of $T^*\mathbf{C} \otimes T\mathbf{C}$ which vanish on edges of \mathbf{C} and which have the property that

$$j \circ \alpha = -\alpha \circ j$$

The space $\Gamma^{0,1}(T\mathbf{C} \otimes T^*\mathbf{C})$ may be regarded as the space of infinitesimal variations of almost complex structure on a fixed domain \mathbf{C} . The action of j on the left makes $\Gamma^{0,1}(T\mathbf{C} \otimes T^*\mathbf{C})$ into a complex vector space.

To obtain $T_f\mathcal{M}^{st}$, we must quotient $\Gamma(T\mathbf{C}) \times \Gamma^{0,1}(T\mathbf{C} \otimes T^*\mathbf{C})$ by those infinitesimal variations which come from reparametrising a given curve.

Given a $C^{\infty,1}$ section v of $T\mathbf{C}$, the change of j under the flow of the vector field v , $L_v j$ is in $\Gamma^{0,1}(T\mathbf{C} \otimes T^*\mathbf{C})$. In fact, $L_v j$ may be regarded as $2j \circ \bar{\partial}v$.

Lemma 2.11. *$L_v j$ may be regarded as $2j \circ \bar{\partial}v$ in the following sense: Let ∇ be any holomorphic connection on $T\mathbf{C}$, then*

$$L_v j = j \circ \nabla v - \nabla v \circ j$$

In particular, in local holomorphic coordinates

$$L_v j = 2j \circ \bar{\partial}v$$

Proof:

If e is any holomorphic vectorfield, then $j \circ \nabla e - (\nabla e) \circ j = 0$, so $j \circ \nabla h e - (\nabla h e) \circ j = j \circ 2(\bar{\partial}h)e$. Therefore, in holomorphic coordinates where v may be considered as a complex function,

$$j \circ \nabla v - \nabla j v = j \circ 2\bar{\partial}v$$

Now we may calculate in holomorphic coordinates, where $j = \partial_y \otimes dx - \partial_x \otimes dy$. Write $v = v_1 \partial_x + v_2 \partial_y$. Then

$$\begin{aligned} L_v j &= -(\partial_y v_1 \partial_x + \partial_y v_2 \partial_y) \otimes dx + \partial_y \otimes (\partial_x v_1 dx + \partial_y v_1 dy) \\ &\quad + (\partial_x v_1 \partial_x + \partial_x v_2 \partial_y) \otimes dy - \partial_x \otimes (\partial_x v_2 dx + \partial_y v_2 dy) \\ &= (\partial_y v_1 + \partial_x v_2) (-\partial_x \otimes dx + \partial_y \otimes dy) \\ &\quad + (\partial_x v_1 - \partial_y v_2) (\partial_y \otimes dx + \partial_x \otimes dy) \end{aligned}$$

On the other hand,

$$\begin{aligned}
j \circ 2\bar{\partial}v &= j \circ dv - dv \circ j \\
&= \partial_x v_1 \partial_y \otimes dx + \partial_y v_1 \partial_y \otimes dy - \partial_x v_2 \partial_x \otimes dx - \partial_y v_2 \partial_x \otimes dy \\
&\quad + \partial_x v_1 \partial_x \otimes dy - \partial_y v_1 \partial_x \otimes dx + \partial_x v_2 \partial_y \otimes dy - \partial_y v_2 \partial_y \otimes dx \\
&= L_v j
\end{aligned}$$

□

Lemma 2.11 implies that $L_{jv}j = j \circ L_v j$, so the corresponding map

$$\begin{array}{ccc}
\Gamma(T\mathbf{C}) & \longrightarrow & \Gamma^{0,1}(T\mathbf{C} \otimes T^*\mathbf{C}) \\
v & \mapsto & L_v j
\end{array}$$

is complex linear. The map

$$\begin{array}{ccc}
h : \Gamma(T\mathbf{C}) & \longrightarrow & \Gamma^{0,1}(T\mathbf{C} \otimes T^*\mathbf{C}) \times \Gamma(f^*T\mathbf{B}) \\
v & \mapsto & (L_v j, df(v))
\end{array}$$

is therefore complex linear if and only if f is holomorphic. If f is in \mathcal{M}^{st} , then f has no infinitesimal automorphisms and h is injective.

Define $T_f \mathcal{M}^{st}$ to be the quotient of $\Gamma^{0,1}(T\mathbf{C} \otimes T^*\mathbf{C}) \times \Gamma(f^*T\mathbf{B})$ by the image of h .

$$0 \longrightarrow \Gamma(T\mathbf{C}) \xrightarrow{h} \Gamma^{0,1}(T\mathbf{C} \otimes T^*\mathbf{C}) \times \Gamma(f^*T\mathbf{B}) \longrightarrow T_f \mathcal{M}^{st} \longrightarrow 0$$

If f is holomorphic, then $T_f \mathcal{M}^{st}$ is a complex vector space. Otherwise, $T_f \mathcal{M}^{st}$ is a real vector space.

2.5.3. $T_f \mathcal{M}^{st}(\hat{\mathbf{B}})$ and $T_f \mathcal{M}^{st}(\hat{\mathbf{B}})_{\downarrow \mathbf{B}_0}$.

In the case of a family of targets $\hat{\mathbf{B}} \rightarrow \mathbf{B}_0$, define the relative tangent space $T_f \mathcal{M}^{st}(\hat{\mathbf{B}})_{\downarrow \mathbf{B}_0}$ to be equal to $T_f \mathcal{M}^{st}(\mathbf{B})$ where \mathbf{B} is the member of the family $\hat{\mathbf{B}}$ which contains the image of f . There is a defining short exact sequence

$$0 \xrightarrow{h} \Gamma(T\mathbf{C}) \longrightarrow \Gamma^{0,1}(T\mathbf{C} \otimes T^*\mathbf{C}) \times \Gamma(f^*T_{vert}\hat{\mathbf{B}}) \longrightarrow T_f \mathcal{M}^{st}(\hat{\mathbf{B}})_{\downarrow \mathbf{B}_0} \longrightarrow 0$$

We may define the tangent space $T_f \mathcal{M}^{st}(\hat{\mathbf{B}})$ similarly to the case of a single target, except we shall use the notation $\Gamma_{\mathbf{B}_0}(f^*T\hat{\mathbf{B}})$ to denote sections of $f^*T\hat{\mathbf{B}}$ which are constant when composed with the derivative of the map $\hat{\mathbf{B}} \rightarrow \mathbf{B}_0$. We use $\Gamma_{\mathbf{B}_0}(f^*T\hat{\mathbf{B}})$ instead of $\Gamma(f^*T\hat{\mathbf{B}})$ because we are interested in infinitesimal variations of f as a curve which is contained in a fiber of $\hat{\mathbf{B}} \rightarrow \mathbf{B}_0$ instead of simply a map to $\hat{\mathbf{B}}$.

$$0 \xrightarrow{h} \Gamma(T\mathbf{C}) \longrightarrow \Gamma^{0,1}(T\mathbf{C} \otimes T^*\mathbf{C}) \times \Gamma_{\mathbf{B}_0}(f^*T\hat{\mathbf{B}}) \longrightarrow T_f \mathcal{M}^{st}(\hat{\mathbf{B}}) \longrightarrow 0$$

2.5.4. Derivatives.

Given any family of curves \hat{f} in \mathcal{M}^{st} containing f and any vector v in $T_f \mathbf{F}(\hat{f})$, we may define an element of $T_f \mathcal{M}^{st}$ by differentiating \hat{f} in the direction of v as follows: Lift v to a $C^{\infty,1}$ section v' of $T\mathbf{C}(\hat{f})$ restricted to $\mathbf{C}(f) \subset \mathbf{C}(\hat{f})$. The fiberwise almost complex structure j is a section of $T_{vert}^*\mathbf{C}(\hat{f}) \otimes T_{vert}\mathbf{C}(\hat{f})$. The Lie derivative of j with respect to any lifted vectorfield is again a section of $T_{vert}^*\mathbf{C}(\hat{f}) \otimes T_{vert}\mathbf{C}(\hat{f})$, because the flow of lifted vectorfields respects the fibers of $\mathbf{C}(\hat{f}) \rightarrow \mathbf{F}(\hat{f})$. The flow of any vectorfield which vanishes on a fiber preserves that fiber, therefore $L_{v'}j$ is well defined, even though we have only defined v' as a section of $T\mathbf{C}(\hat{f})$ restricted

to $\mathbf{C}(f)$. Because v' is locally equal to a j -preserving vectorfield plus a section of $T\mathbf{C}(f)$, Lemma 2.11 implies that $L_{v'}j$ is a section of $\Gamma^{0,1}(T^*\mathbf{C}(f) \otimes T\mathbf{C}(f))$. Therefore,

$$(L_{v'}j, df(v'))$$

gives an element of $\Gamma^{0,1}(T\mathbf{C}(f) \otimes T^*\mathbf{C}(f)) \times \Gamma_{\mathbf{B}_0}(f^*T\hat{\mathbf{B}})$. As any other lift v'' of v will differ from v' by a section of $T\mathbf{C}(f)$, $(L_{v''}j, df(v''))$ will differ from $(L_{v'}j, df(v'))$ by a vector in the image of h . Therefore, $(L_{v'}j, df(v'))$ gives a well defined element of $T_f\mathcal{M}^{st}$.

Therefore for any $f \in \hat{f}$, we have a well defined linear map

$$T_f\mathbf{F}(\hat{f}) \longrightarrow T_f\mathcal{M}^{st}$$

Obviously, this map restricts to a well defined linear map

$$T_f\mathbf{F}(\hat{f})|_{\mathbf{B}_0} \longrightarrow T_f\mathcal{M}^{st}(\hat{\mathbf{B}})|_{\mathbf{B}_0}$$

Lemmas 2.12 and 2.13 below imply that as a set, $T_f\mathcal{M}^{st}$ may be also regarded as the quotient of the space of families of curves parametrized by \mathbb{R} containing f at 0 by an equivalence relation generated by setting two such families to be equivalent if they are tangent at 0 within a two dimensional family of curves. It follows that given any $C^{\infty,1}$ map

$$\Phi : \mathcal{M}^{st} \longrightarrow \mathbf{A}$$

there is an induced linear derivative map

$$T_f\Phi : T_f\mathcal{M}^{st} \longrightarrow T_{\Phi(f)}\mathbf{A}$$

Lemma 2.12. *Given any curve f in $\mathcal{M}^{st}(\hat{\mathbf{B}})$ and vector in $T_f\mathcal{M}^{st}(\hat{\mathbf{B}})$, there exists a family of curves parametrized by \mathbb{R} with that vector in its image.*

Proof:

We must construct a family with a given derivative. To avoid any issues with the precise nature of almost complex structures at edges of a curve, we can reduce to the case that the variation in almost complex structure on the domain \mathbf{C} is described by a section of $\Gamma^{0,1}(T\mathbf{C} \otimes T^*\mathbf{C})$ which vanishes in a neighborhood of all edges of \mathbf{C} .

Consider a neighborhood of a edge of the domain \mathbf{C} of f . This may be identified with an open subset of $\mathbf{T}_{[0,l]}^1$ or $\mathbf{T}_{[0,\infty)}^1$ with its standard complex structure. A vector field v may then be written in standard coordinates as $C^{\infty,1}$ functions times the real and imaginary parts of $\tilde{z}\frac{\partial}{\partial \tilde{z}}$. As noted in Lemma 2.11, in these coordinates L_vj is just $2j$ times the standard $\bar{\partial}$ operator. Therefore, Theorem 3.8 on page 25 implies that given any section in $\Gamma^{0,1}(T\mathbf{C} \otimes T^*\mathbf{C})$, there exists a $C^{\infty,1}$ vectorfield v so that L_vj is equal to the given section in a neighborhood of each edge.

Given any section α in $\Gamma^{0,1}(T\mathbf{C} \otimes T^*\mathbf{C})$ which vanishes in a neighborhood of edges of \mathbf{C} , it is straightforward to construct a family of almost complex structures j_t on \mathbf{C} so that j_t is the original almost complex structure when $t = 0$ and near edges, and $\frac{\partial}{\partial t}j_t = \alpha$ at $t = 0$. It is also straightforward to construct a $C^{\infty,1}$ family of maps f_t from \mathbf{C} to $\hat{\mathbf{B}} \rightarrow \mathbf{B}_0$ so that $\frac{\partial}{\partial t}f_t$ at $t = 0$ is any given $C^{\infty,1}$ section of $f^*T\hat{\mathbf{B}}$ which projects to be a constant map to $T\mathbf{B}_0$. It follows that given any vector in $T_f\mathcal{M}^{st}$, there exists a family of curves parametrized by \mathbb{R} with the given derivative. □

Lemma 2.13. *Suppose that \hat{f}_1 and \hat{f}_2 are two families parametrized by \mathbb{R} which contain f and which have the same image in $T_f\mathcal{M}^{st}$. Then there exists another*

family \hat{f}_0 containing f and two 2-dimensional families \hat{g}_i with given maps $\hat{f}_0 \rightarrow \hat{g}_i$ and $\hat{f}_i \rightarrow \hat{g}_i$ so that the maps

$$\mathbf{F}(\hat{f}_0) \rightarrow \mathbf{F}(\hat{g}_i)$$

$$\mathbf{F}(\hat{f}_i) \rightarrow \mathbf{F}(\hat{g}_i)$$

are tangent at f .

Proof: The reason that an extra family \hat{f}_0 is needed and the heart of the technical problem that must be overcome by this proof is the following observation: Given a smooth section of an infinite dimensional vectorbundle over the real line, there may not exist a finite dimensional sub vectorbundle containing the given smooth section.

Let t indicate the coordinate parametrizing $\mathbf{F}(\hat{f}_i)$, and suppose that f is the curve over 0. The following claim is a version of Hadamard's lemma:

Claim 2.14. *If a $C^{\infty,1}$ section ν of a vector bundle over $\mathbf{C}(\hat{f}_i)$ vanishes at $t = 0$, then $\nu = t\nu'$ where ν' is also a $C^{\infty,1}$ section. If ν vanishes on all edges of curves in $\mathbf{C}(\hat{f}_i)$, then ν' does too.*

To prove Claim 2.14, choose a connection ∇ on our vector bundle, and a lift v of $\frac{\partial}{\partial t}$ to a $C^{\infty,1}$ vectorfield on $\mathbf{C}(\hat{f}_i)$. The flow of v identifies $\mathbf{C}(\hat{f}_i)$ with $\mathbf{C}(f) \times \mathbb{R}$ and together with ∇ allows us to trivialize the vectorbundle in the \mathbb{R} direction. Then we may write

$$\nu(z, t) = \int_0^1 (\nabla_{tv}\nu)(z, ts) ds = t \int_0^1 (\nabla_v\nu)(z, ts) ds$$

$\nabla_v\nu$ is a $C^{\infty,1}$ section of our vectorbundle which vanishes on edges of curves in $\mathbf{C}(\hat{f}_i)$ if ν does, therefore

$$\nu'(z, t) := \int_0^1 (\nabla_v\nu)(z, ts) ds$$

is a $C^{\infty,1}$ section of our vectorbundle which vanishes on edges of curves in $\mathbf{C}(\hat{f}_i)$ if ν does. This completes the proof of Claim 2.14.

Claim 2.14 implies that if a section of a vector bundle over $\mathbf{C}(\hat{f}_i)$ vanishes at $t = 0$ to order $n - 1$, but has nonvanishing n th derivative, then it is equal to t^n times a $C^{\infty,1}$ section which does not vanish at 0.

It suffices to prove this lemma for a neighborhood of f in \hat{f}_i , so we shall repeatedly assume \hat{f}_i is small enough as needed.

If the domain of f is not stable, then we can choose some (not necessarily closed or connected) codimension 2 surface S in \mathbf{B} so that f is transverse to S , and $\mathbf{C}(f)$ with the extra marked points in $f^{-1}S$ is stable. Then a neighborhood of f in \hat{f}_i will remain transverse to S . For simplicity, we shall assume that \hat{f}_i remains everywhere transverse to S .

As explained in section 4.1, there is a unique $C^{\infty,1}$ map $s_i : \mathbb{R} \rightarrow \mathcal{M}(pt)$ so that the pullback of the universal curve over $\mathcal{M}(pt)$ is $\mathbf{C}(\hat{f}_i)$ with extra edges at each of the points in $\hat{f}_i^{-1}(S)$. Moreover, it is proved in section 4.1 that we may locally represent $\mathcal{M}(pt)$ as the quotient of a given family of stable curves by a finite group of automorphisms. These two maps $s_i : \mathbb{R} \rightarrow \mathcal{M}(pt)$ are tangent at 0. Choose another map $s_0 : \mathbb{R} \rightarrow \mathcal{M}(pt)$ which is tangent to them at 0, but not equal to either of them to second order. Then Claim 2.14 implies that there exist two smooth maps $\tilde{s}_i : \mathbb{R}^2 \rightarrow \mathcal{M}(pt)$ so that $s_i(t)$ is equal to $\tilde{s}_i(t, 0)$, and $s_0(t) = \tilde{s}_i(t, t^2)$.

Define the domain $\mathbf{C}(\hat{g}_i)$ of \hat{g}_i to be the pullback of the universal curve by \tilde{s}_i with extra edges (which corresponded to $\hat{f}_i^{-1}(S)$) removed (this operation replaces

a neighborhood of an edge with its smooth part - replacing an open subset of \mathbf{T}_1^1 with the corresponding open subset $\mathbb{C} = [\mathbf{T}_1^1]$.) Around the image of f in $\mathcal{M}(pt)$, we may identify all the domains of curves in the universal curve over $\mathcal{M}(pt)$, and make the identifications holomorphic in a neighborhood of edges. Therefore, restricted to a neighborhood of $0 \in \mathbb{R}^2$, we can regard $\mathbf{C}(\hat{g}_i)$, and hence $\mathbf{C}(\hat{f}_i)$ to be given by a family of almost complex structures on a fixed domain, and regard $\mathbf{C}(\hat{f}_i)$ to be identical, but have different almost complex structures for $i = 0, 1, 2$. With these identifications, the fact that \hat{f}_i are tangent for $i = 1, 2$ implies that the maps \hat{f}_i are equal to first order restricted to $\mathbf{C}(f)$. We may therefore choose a map \hat{f}_0 sending $\hat{f}_i^{-1}(S)$ to S which is equal \hat{f}_i to first order at $\mathbf{C}(f)$, but not equal to either of them to second order at $\mathbf{C}(f)$. Claim 2.14 then implies that there exist families $\hat{g}_i : \mathbf{C}(\hat{g}_i) \rightarrow \mathbf{B}$ so \hat{f}_i is the restriction of \hat{g}_i to $\mathbf{C}(\hat{f}_i)$, and \hat{f}_0 is the restriction of \hat{g}_i to $\mathbf{C}(\hat{f}_0)$. \square

Lemmas 2.12 and 2.13 allow us to differentiate any $C^{\infty,1}$ map $\Phi : \mathcal{M}^{st} \rightarrow \mathbf{X}$ to obtain a linear map

$$T_f \Phi : T_f \mathcal{M}^{st} \rightarrow T_{\Phi(f)} \mathbf{X}$$

defined so that for all $C^{\infty,1}$ families of curves \hat{f} containing f , the diagram

$$\begin{array}{ccc} T_f \mathcal{M}^{st} & \xrightarrow{T_f \Phi} & T_{\Phi(f)} \mathbf{X} \\ \uparrow & \searrow T_f \Phi_{\hat{f}} & \\ T_f \mathbf{F}(\hat{f}) & & \end{array}$$

commutes, where $\Phi_{\hat{f}} : \mathbf{F}(\hat{f}) \rightarrow \mathbf{X}$ is the map induced by Φ . Lemma 2.12 implies that $T_f \Phi$ is unique, and Lemma 2.13 implies that it is well defined.

2.5.5. \mathbb{R} -nil vectors.

Recall that the integral vectors of an exploded manifold are the vectors v so that $\tilde{z}^{-1}v\tilde{z}$ is always an integer for any exploded function \tilde{z} . Such vectors always act as zero derivations on smooth or $C^{\infty,1}$ functions.

Definition 2.15. *A \mathbb{R} -nil vector v on an exploded manifold is a vector which acts as a zero derivation on any $C^{\infty,1}$, \mathbb{R} -valued function.*

There is a canonical complex structure on the \mathbb{R} -nil vectors at a point so that $(Jv)(\tilde{z}) = i(v\tilde{z})$. The \mathbb{R} -nil vectors at a point are always the complex linear span of the integral vectors. Clearly derivatives always send \mathbb{R} -nil vectors to \mathbb{R} -nil vectors, and such derivative maps are always complex with respect to the canonical complex structure on \mathbb{R} -nil vectors.

The bundle of \mathbb{R} -nil vectors on a strata of an exploded manifold have a canonical flat connection, which is the connection which preserves the canonical complex structure and the lattice of integral vectors— so a constant \mathbb{R} -nil vector field may be written as some sum of complex numbers times integral vector fields.

There is a similar notion of integral and \mathbb{R} -nil vectors on $T_f \mathcal{M}^{st}$ and $T_f \mathcal{M}^{st} \downarrow_{\mathbf{B}_0}$.

Definition 2.16. *A vector v in $T_f \mathcal{M}^{st}$ is \mathbb{R} -nil or integral if there exists a family of curves \hat{f} containing f and a \mathbb{R} -nil or integral vector in $T_f \mathbf{F}(\hat{f})$ with image $v \in T_f \mathcal{M}^{st}$.*

As the canonical complex structure on \mathbb{R} -nil vectors is compatible with all exploded maps, it follows that there is a canonical complex structure on the \mathbb{R} -nil vectors within $T_f\mathcal{M}^{st}$.

2.5.6. $T_{\hat{f}}\mathcal{M}^{st}$.

For a family of curves \hat{f} , it is not clear that the vector spaces $T_f\mathcal{M}^{st}$ for all f in \hat{f} fit together to form a vector bundle. On the other hand, there is a natural tangent sheaf, $T_{\hat{f}}\mathcal{M}^{st}$ which should be regarded as giving first order deformations of \hat{f} parametrized by $\mathbf{F}(\hat{f})$. This tangent sheaf is defined by the short exact sequence

$$(1) \quad 0 \rightarrow \Gamma(T_{vert}\mathbf{C}(\hat{f})) \rightarrow \Gamma^{0,1}(T_{vert}\mathbf{C}(\hat{f}) \otimes T_{vert}^*\mathbf{C}(\hat{f})) \times \Gamma_{\mathbf{B}_0}(\hat{f}^*T\hat{\mathbf{B}}) \rightarrow T_{\hat{f}}\mathcal{M}^{st}(\hat{\mathbf{B}}) \rightarrow 0$$

and in the relative case by the short exact sequence

$$(2) \quad 0 \rightarrow \Gamma(T_{vert}\mathbf{C}(\hat{f})) \rightarrow \Gamma^{0,1}(T_{vert}\mathbf{C}(\hat{f}) \otimes T_{vert}^*\mathbf{C}(\hat{f})) \times \Gamma(\hat{f}^*T_{vert}\hat{\mathbf{B}}) \rightarrow T_{\hat{f}}\mathcal{M}^{st}(\hat{\mathbf{B}}) \downarrow_{\mathbf{B}_0} \rightarrow 0$$

Again, $\Gamma^{0,1}(T_{vert}\mathbf{C}(\hat{f}) \otimes T_{vert}^*\mathbf{C}(\hat{f}))$ indicates $C^{\infty,1}$ sections of $(T_{vert}\mathbf{C}(\hat{f}) \otimes T_{vert}^*\mathbf{C}(\hat{f}))$ which vanish on edges of curves and which anti commute with j . These sections represent infinitesimal variations of complex structure on the domain. $\Gamma_{\mathbf{B}_0}(\hat{f}^*T\hat{\mathbf{B}})$ indicates $C^{\infty,1}$ sections of $\hat{f}^*T\hat{\mathbf{B}}$ which are the lift of some section over $\mathbf{F}(\hat{f})$ of the pullback of $T\mathbf{B}_0$.

The infinitesimal reparametrizations of $\mathbf{C}(\hat{f})$ which fix the parametrization of $\mathbf{F}(\hat{f})$ are represented by $C^{\infty,1}$ sections of $T_{vert}\mathbf{C}(\hat{f})$. The action of reparametrization is again

$$\begin{array}{ccc} \Gamma(T_{vert}\mathbf{C}(\hat{f})) & \longrightarrow & \Gamma^{0,1}(T_{vert}\mathbf{C}(\hat{f}) \otimes T_{vert}^*\mathbf{C}(\hat{f})) \times \Gamma(\hat{f}^*T_{vert}\hat{\mathbf{B}}) \\ v & \mapsto & (L_v j, d\hat{f}(v)) \end{array}$$

$T_{\hat{f}}\mathcal{M}^{st}$ is a sheaf of $C^{\infty,1}(\mathbf{F}(\hat{f}))$ -modules. Note that if f is a curve in \hat{f} , we can restrict any section of $T_{\hat{f}}\mathcal{M}^{st}$ to give a tangent vector in $T_f\mathcal{M}^{st}$, and more generally, given any map $\hat{g} \rightarrow \hat{f}$, there is an induced map $T_{\hat{f}}\mathcal{M}^{st} \rightarrow T_{\hat{g}}\mathcal{M}^{st}$. The sheaf $T_{\hat{f}}\mathcal{M}^{st}$ should be regarded the pullback of the tangent sheaf of \mathcal{M}^{st} . When the family \hat{f} has bubbling or node formation behavior, it is not clear that $T_{\hat{f}}\mathcal{M}^{st}$ represents sections of a vector bundle over $\mathbf{F}(\hat{f})$, however any subsheaf of $T_{\hat{f}}\mathcal{M}^{st}$ which is locally free and finitely generated over $C^{\infty,1}(\mathbf{F})$ may be regarded as sections of a finite dimensional sub-vectorbundle of $T_{\hat{f}}\mathcal{M}^{st}$.

Given any family \hat{h} of curves containing \hat{f} as an embedded sub family, there is a natural map to $T_{\hat{f}}\mathcal{M}^{st}$ from $C^{\infty,1}$ sections of the restriction of $T\mathbf{F}(\hat{h})$ to $\mathbf{F}(\hat{f})$. In particular, given any such section v of $T\mathbf{F}(\hat{h})|_{\mathbf{F}(\hat{f})}$, we may choose a lift \tilde{v} of v to a section of the restriction of $T\mathbf{C}(\hat{h})$ to $\mathbf{C}(\hat{f})$. Then $(L_{\tilde{v}}j, d\hat{h}(\tilde{v}))$ is contained in $\Gamma^{0,1}(T_{vert}\mathbf{C}(\hat{f}) \otimes T_{vert}^*\mathbf{C}(\hat{f})) \times \Gamma_{\mathbf{B}_0}(\hat{f}^*T\hat{\mathbf{B}})$. The choice of lift of \tilde{v} is determined up to a choice of vertical vector field, so $(L_{\tilde{v}}j, d\hat{h}(\tilde{v}))$ projects to a section of $T_{\hat{f}}\mathcal{M}^{st}$ which is well defined independent of our choice of lift of v .

The following lemma shows that all sections of $T_{\hat{f}}\mathcal{M}^{st}$ may be constructed in this way.

Lemma 2.17. *Given any section v of $T_{\hat{f}}\mathcal{M}^{st}$, there is a one dimensional family \hat{f}_t of families of curves parametrized by $\mathbf{F}(\hat{f})$ with derivative at $t = 0$ equal to v .*

Proof:

As noted in Claim 2.11, in holomorphic coordinates, $L_w j$ is $2j$ times the standard $\bar{\partial}$ operator applied to w . Around any curve f in \hat{f} , Theorem 3.8 on page 25 implies that we may construct a vertical vector field w so that $L_w j$ is equal to any given section of $\Gamma^{0,1}(T_{vert}\mathbf{C}(\hat{f}) \otimes T_{vert}^*\mathbf{C}(\hat{f}))$ on a neighborhood in $\mathbf{C}(\hat{f})$ of all the edges of $\mathbf{C}(f)$. Piecing together such vectorfields for different f using a partition of unity from $\mathbf{F}(\hat{f})$ gives a globally defined section w of $T_{vert}\mathbf{C}(\hat{f})$ so that $L_w j$ is equal to the given section of $\Gamma^{0,1}(T_{vert}\mathbf{C}(\hat{f}) \otimes T_{vert}^*\mathbf{C}(\hat{f}))$ on a neighborhood of the edges of all the curves in $\mathbf{C}(\hat{f})$.

We may therefore reduce to the case that our section v of $T_{\hat{f}}\mathcal{M}^{st}$ is equal to (θ, r) where θ is a section of $\Gamma^{0,1}(T_{vert}\mathbf{C}(\hat{f}) \otimes T_{vert}^*\mathbf{C}(\hat{f}))$ which vanishes on a neighborhood of all edges of curves in $\mathbf{C}(\hat{f})$, and r is a $C^{\infty,1}$ section of $\hat{f}^*T_{vert}\hat{\mathbf{B}}$.

Extend j to a family j_t of almost complex structures on $\mathbf{C}(\hat{f})$ with derivative at 0 equal to θ , and extend \hat{f} to a family of maps \hat{f}_t with derivative at 0 equal to r . The family of maps f_t with domain $(\mathbf{C}(\hat{f}), j_t)$ is the required deformation of \hat{f} with derivative at 0 equal to v . □

2.6. $D\bar{\partial}$.

In this section, we shall construct the linearization $D\bar{\partial}$ of the $\bar{\partial}$ operator at holomorphic curves f .

To define a linearization of the $D\bar{\partial}$ operator at a non holomorphic curve f , a connection is required on the sheaf \mathcal{Y} over \mathcal{M}^{st} . It is not clear to me that such a connection always exists globally.

Given a $C^{\infty,1}$ family of curves \hat{f} in $\hat{\mathbf{B}}$, $\bar{\partial}\hat{f}$ may be regarded as a section of the vector bundle

$$Y(\hat{f}) := \left(T_{vert}^*\mathbf{C}(\hat{f}) \otimes \hat{f}^*T_{vert}\hat{\mathbf{B}} \right)^{(0,1)}$$

over $\mathbf{C}(\hat{f})$. This section $\bar{\partial}\hat{f}$ vanishes on all edges of curves in $\mathbf{C}(\hat{f})$.

Suppose that f is a curve in \hat{f} so that $\bar{\partial}f = 0$. Then we may differentiate $\bar{\partial}\hat{f}$ restricted to $\mathbf{C}(\hat{f})$ using any $C^{\infty,1}$ section v' of $T\mathbf{C}(\hat{f})$ restricted to $\mathbf{C}(f)$ which projects to a single vector v in $T_f\mathbf{F}(\hat{f})$. Because $\bar{\partial}\hat{f}$ vanishes on $\mathbf{C}(f)$, the result of this differentiation does not depend on the choice of lift v' of v . Because $\bar{\partial}\hat{f}$ vanishes on edges of $\mathbf{C}(f)$, the result of this differentiation vanishes on all edges of curves in $\mathbf{C}(\hat{f})$, so it is a section in $\mathcal{Y}(f)$. We therefore obtain a linear map

$$D\bar{\partial} : T_f\mathbf{F} \longrightarrow \mathcal{Y}(f)$$

Given any commutative diagram

$$\begin{array}{ccc} f & \longrightarrow & \hat{f} \\ & \searrow & \downarrow \\ & & h \end{array}$$

the following diagram commutes

$$\begin{array}{ccc} T_f\mathbf{F}(\hat{f}) & \xrightarrow{D\bar{\partial}} & T_f\mathcal{M}^{st} \\ \downarrow & \nearrow D\bar{\partial} & \\ T_f\mathbf{F}(h) & & \end{array}$$

Therefore, Lemmas 2.12 and 2.13 imply that the maps $D\bar{\partial}$ must come from a linear map

$$T_f \mathbf{F} \longrightarrow T_f \mathcal{M}^{st} \xrightarrow{D\bar{\partial}} \mathcal{Y}(f)$$

An elementary extension of the calculation of the linearized $\bar{\partial}$ operator in [13] gives a formula for $D\bar{\partial}$ in terms of $\Gamma^{0,1}(T_{vert}^* \mathbf{C}(f) \otimes T_{vert}^* \mathbf{C}(f)) \times \Gamma_{\mathbf{B}_0}(f^* T\hat{\mathbf{B}})$.

Recall that both $\mathcal{Y}(f)$ and $T_f \mathcal{M}^{st} \downarrow_{\mathbf{B}_0} \subset T_f \mathcal{M}^{st}$ are complex at holomorphic curves f . The restriction of $D\bar{\partial}$ to $T_f \mathcal{M}^{st} \downarrow_{\mathbf{B}_0}$ is not usually \mathbb{C} -linear unless J is integrable. We shall see that $D\bar{\partial}$ has a finite dimensional kernel and cokernel throughout the linear homotopy of $D\bar{\partial}$ to its complex linear part. This allows us to canonically orient the kernel of $D\bar{\partial}$ relative to the cokernel. This homotopy is used in section 8 to construct orientations and a kind of almost complex structure on embedded Kuranishi structures.

2.7. Embedded Kuranishi structures.

If $D\bar{\partial}$ is surjective at a holomorphic curve f , then we shall show that the moduli stack of holomorphic curves close to f is well behaved, and may be represented by the quotient of some $C^{\infty,1}$ family of curves \hat{f} by a group of automorphisms. If $D\bar{\partial}$ is not surjective at f , we shall choose a nice subsheaf V of \mathcal{Y} in a neighborhood of f so that the moduli stack of curves with $\bar{\partial}$ in V is well behaved. Below is the definition of a subsheaf V of \mathcal{Y} .

Definition 2.18. *Let \mathcal{U} be a substack of \mathcal{M}^{st} . A subsheaf V of \mathcal{Y} on \mathcal{U} is an assignment to each family \hat{f} of curves in \mathcal{U} a subsheaf $V(\hat{f}) \subset \mathcal{Y}(\hat{f})$ so that given any map of families of curves $\hat{f} \rightarrow \hat{g}$, the induced map $\mathcal{Y}(\hat{g}) \rightarrow \mathcal{Y}(\hat{f})$ restricts to give a map*

$$V(\hat{g}) \longrightarrow V(\hat{f})$$

so that $V(\hat{f})$ is equal to the sheaf of $C^{\infty,1}(\mathbf{F}(\hat{f}))$ -modules generated by the image of $V(\hat{g})$.

We shall want our subsheaves V of \mathcal{Y} to be pulled back from nice geometrically defined sheaves. For example, if f is embedded and has a stable domain, we may choose to pull back V from a sheaf defined over $\bar{M}_{g,n} \times \hat{\mathbf{B}}$. The following defines what we shall mean by pullback.

Definition 2.19. *Given a family $\hat{\mathbf{A}} \rightarrow \mathbf{X}$ of exploded manifolds with a fiberwise almost complex structure, define*

$$\Gamma^{0,1}(T_{vert}^* \hat{\mathbf{A}} \otimes T_{vert} \hat{\mathbf{B}})$$

to be a sheaf of $C^{\infty,1}(\mathbf{X})$ -modules on \mathbf{X} with global sections consisting of anti-holomorphic, $C^{\infty,1}$ sections of $T_{vert}^ \hat{\mathbf{A}} \otimes T_{vert} \hat{\mathbf{B}}$ over $\hat{\mathbf{A}} \times \hat{\mathbf{B}}$ which vanish on integral vectors within $T_{vert} \hat{\mathbf{A}}$, and with sections over $U \subset \mathbf{X}$ consisting of the same thing with $\hat{\mathbf{A}}$ replaced by the inverse image of U in $\hat{\mathbf{A}}$.*

Given a family of curves \hat{f} in $\hat{\mathbf{B}}$ and a fiberwise holomorphic map

$$\begin{array}{ccc} \mathbf{C}(\hat{f}) & \xrightarrow{\psi} & \hat{\mathbf{A}} \\ \downarrow & & \downarrow \\ \mathbf{F}(\hat{f}) & \longrightarrow & \mathbf{X} \end{array}$$

sections of $T_{vert}^ \hat{\mathbf{A}}$ may be pulled back to sections of $T_{vert}^* \mathbf{C}(\hat{f})$ using ψ^* . As ψ is fiberwise holomorphic, the pullback map, ψ^* , is complex. Any section which*

vanishes on integral vectors of $T_{\text{vert}}\hat{\mathbf{A}}$ pulls back to a section which vanishes on integral vectors of $T_{\text{vert}}\mathbf{C}(\hat{f})$.

A section of $T_{\text{vert}}\hat{\mathbf{B}}$ also pulls back to give a section of $\hat{f}^*T_{\text{vert}}\hat{\mathbf{B}}$. This pullback map is also complex, therefore there is an induced complex map

$$\Gamma^{0,1}(T_{\text{vert}}^*\hat{\mathbf{A}} \otimes T_{\text{vert}}\hat{\mathbf{B}}) \longrightarrow \mathcal{Y}(\hat{f}) := \Gamma^{0,1}(T_{\text{vert}}^*\mathbf{C}(\hat{f}) \otimes f^*T_{\text{vert}}\hat{\mathbf{B}})$$

Say that the pullback of a section or subsheaf of $\Gamma^{0,1}(T_{\text{vert}}^*\hat{\mathbf{A}} \otimes T_{\text{vert}}\hat{\mathbf{B}})$ to $\mathcal{Y}(\hat{f})$ is the image of the section or subsheaf under the above map.

The following defines what we mean by a ‘nice’ subsheaf V of \mathcal{Y} .

Definition 2.20. Say that a subsheaf V of \mathcal{Y} on \mathcal{U} is simply generated if there exists a fiberwise holomorphic map

$$\begin{array}{ccc} \mathcal{U}^{+1} & \longrightarrow & \hat{\mathbf{A}}/G \\ \downarrow & & \downarrow \\ \mathcal{U} & \longrightarrow & \mathbf{X}/G \end{array}$$

with an effective G action in the sense of Definition 2.5, and sections v_1, \dots, v_n of $\Gamma^{0,1}(T_{\text{vert}}^*\hat{\mathbf{A}} \otimes T_{\text{vert}}\hat{\mathbf{B}})$ so that

- (1) For any family of curves \hat{f} in \mathcal{U} , the pullback in the sense of Definition 2.19 of v_1, \dots, v_n to $\mathcal{Y}(\hat{f})$ are linearly independent and generate $V(\hat{f})$ as a sheaf of $C^{\infty,1}(\mathbf{F}(\hat{f}'))$ -modules. (Recall that as in Definition 2.5, \hat{f}' indicates the G -fold cover of \hat{f} whose domain maps to \mathbf{A} .)
- (2) The sheaf of $C^{\infty,1}(\mathbf{X})$ modules over \mathbf{X} generated by v_1, \dots, v_n is G -invariant.

Theorem 6.6 on page 60 states that if V is a simply generated subsheaf of \mathcal{Y} so that for some holomorphic curve f , $V(f)$ is transverse to $D\bar{\partial} : T_f\mathcal{M}^{st}(\hat{\mathbf{B}}) \downarrow_{\mathbf{B}_0} \longrightarrow \mathcal{Y}(f)$, then there exists an open neighborhood \mathcal{O} of f and a $C^{\infty,1}$ family of curves \hat{f} with automorphism group G so that \hat{f}/G represents the substack of curves h in \mathcal{O} with the property that $\bar{\partial}h \in V(h)$. The substack of \hat{f}/G corresponding to the holomorphic curves in \hat{f}/G therefore represents the moduli stack of holomorphic curves in \mathcal{O} . Therefore we may regard the moduli stack of holomorphic curves in \mathcal{O} as represented by the quotient by G of the intersection of the section $\bar{\partial}\hat{f}$ of $V(\hat{f}) \longrightarrow \mathbf{F}(\hat{f})$ with the zero section.

Definition 2.21. Say that a subsheaf V of \mathcal{Y} is complex if $V(f)$ is a complex linear subspace of $\mathcal{Y}(f)$ for all curves f within the domain of definition of V .

Recall that if f is holomorphic, $T_f\mathcal{M}^{st} \downarrow_{\mathbf{B}_0}$ is a complex vector space, and there is a well defined linearization of the $\bar{\partial}$ operator

$$D\bar{\partial} : T_f\mathcal{M}^{st} \downarrow_{\mathbf{B}_0} \longrightarrow \mathcal{Y}(f)$$

constructed in section 2.6. This map $D\bar{\partial}$ is not necessarily \mathbb{C} -linear, however we shall show that there is a homotopy from it to its complex linear part, $(1-t)D\bar{\partial} + tD\bar{\partial}^{\mathbb{C}}$ which is transverse to a finite dimensional \mathbb{C} -linear subspace V of $\mathcal{Y}(f)$ for all $t \in [0, 1]$, so

$$K_t(f) := ((1-t)D\bar{\partial} + tD\bar{\partial}^{\mathbb{C}})^{-1}(V)$$

is a family of vector spaces in $T_f\mathcal{M}^{st} \downarrow_{\mathbf{B}_0}$ of some finite dimension.

Definition 2.22. Say that a subsheaf V of \mathcal{Y} is strongly transverse to $D\bar{\partial}$ at a holomorphic curve f if

$$(1-t)D\bar{\partial} + tD\bar{\partial}^{\mathbb{C}} : T_f\mathcal{M}^{st} \downarrow_{\mathbf{B}_0} \longrightarrow \mathcal{Y}(f)$$

is transverse to V for all $t \in [0, 1]$.

Definition 2.23. A Kuranishi chart $(\mathcal{U}, V, \hat{f}/G)$ on \mathcal{M}^{st} is

- an open substack

$$\mathcal{U} \subset \mathcal{M}^{st}$$

- a simply generated complex subsheaf

$$V$$

of \mathcal{Y} defined on \mathcal{U} so that V is strongly transverse to $\bar{\partial}$ at all holomorphic curves in \mathcal{U} ,

- a $C^{\infty,1}$ family of curves \hat{f} in \mathcal{U} with automorphism group G so that
 - (1) \hat{f}/G represents the substack of \mathcal{U} consisting of families of curves \hat{h} with $\bar{\partial}\hat{h}$ a section of $V(\hat{h})$.
 - (2) In the case of $\mathcal{M}^{st}(\hat{\mathbf{B}})$, where $\hat{\mathbf{B}}$ is a family of targets over the exploded manifold \mathbf{B}_0 , the map $\mathbf{F}(\hat{f}) \rightarrow \mathbf{B}_0$ is a submersion.

$\mathbf{F}(\hat{f})$ with the vector bundle $V(\hat{f})$, the section of this vector bundle given by $\bar{\partial}$ and the action of G should be regarded as a concrete version of a Kuranishi chart defined by Fukaya and Ono in [3].

The condition that $\mathbf{F}(\hat{f}) \rightarrow \mathbf{B}_0$ is a submersion is included to ensure that any base change of our family of targets

$$\begin{array}{ccc} \hat{\mathbf{B}}' & \longrightarrow & \hat{\mathbf{B}} \\ \downarrow & & \downarrow \\ \mathbf{B}'_0 & \longrightarrow & \mathbf{B}_0 \end{array}$$

pulls back Kuranishi charts on $\mathcal{M}^{st}(\hat{\mathbf{B}})$ to Kuranishi charts on $\mathcal{M}^{st}(\hat{\mathbf{B}}')$.

The condition of strong transversality to V and the complex structure on V is in the definition of a Kuranishi chart to ensure that the information of the stable almost complex structure defined by Fukaya and Ono in [3] is reflected in our Kuranishi chart. This information may be used to orient a Kuranishi chart, and may also used to define integer counts of holomorphic curves using Fukaya and Ono's method from [4]. In [7], Joyce also outlines a method to use such structure to obtain integrality results for Gromov-Witten invariants. The kind of evaluation maps which are compatible with these integer invariants are the holomorphic submersions defined below:

Definition 2.24. A submersion

$$\begin{array}{ccc} \mathcal{M}^{st} & \xrightarrow{\Phi} & \mathbf{X} \\ \downarrow & & \downarrow \\ \mathbf{B}_0 & \longrightarrow & \mathbf{X}_0 \end{array}$$

is a commutative diagram of $C^{\infty,1}$ maps so that $T_f\Phi : T_f\mathcal{M}^{st} \rightarrow T_{\Phi(f)}\mathbf{X}$ is surjective for all f .

Say that a submersion $\Phi : \mathcal{M}^{st} \rightarrow \mathbf{X}$ is holomorphic if $\mathbf{X} \rightarrow \mathbf{X}_0$ has a fiberwise almost complex structure, and for each holomorphic curve f , the map

$$T_f\Phi : T_f\mathcal{M}^{st} \downarrow_{\mathbf{B}_0} \rightarrow T_{\Phi(f)}\mathbf{X} \downarrow_{\mathbf{X}_0}$$

is complex.

Examples of such holomorphic submersions include the maps ev^{+n} defined in section 4.2, and the usual evaluation map from Gromov-Witten theory which evaluates curves at marked points.

Definition 2.25. *Given a submersion*

$$\Phi : \mathcal{M}^{st} \longrightarrow \mathbf{X}$$

where \mathbf{X} is a finite dimensional exploded manifold or orbifold, say that a Kuranishi chart $(\mathcal{U}, V, \hat{f}/G)$ on \mathcal{M}^{st} is Φ -submersive if the induced map

$$\Phi : \mathbf{F}(\hat{f}) \longrightarrow \mathbf{X}$$

is a submersion, and if for all holomorphic curves f in \hat{f} , $D\bar{\partial}$ restricted to $\ker T_f \Phi$ is strongly transverse to $V(f)$ in the sense that

$$((1-t)D\bar{\partial} + tD\bar{\partial}^{\mathbb{C}})(T_f \mathcal{M}^{st} \downarrow_{\mathbf{B}_0} \cap \ker T_f \Phi)$$

is transverse to $V(f)$ for all t in $[0, 1]$.

Definition 2.26. *Two Kuranishi charts $(\mathcal{U}_1, V_1, \hat{f}_1/G_1)$, $(\mathcal{U}_2, V_2, \hat{f}_2/G_2)$ are compatible if restricted to $\mathcal{U}_1 \cap \mathcal{U}_2$, either V_1 is a subsheaf of V_2 or V_2 is a subsheaf of V_1 .*

Note that if restricted to $\mathcal{U}_1 \cap \mathcal{U}_2$, V_1 is a subsheaf of V_2 , then there is a unique $C^{\infty, 1}$ map

$$\hat{f}_1|_{\mathcal{U}_1 \cap \mathcal{U}_2} \longrightarrow \hat{f}_2/G_2$$

which may be regarded as a transition map between these two compatible Kuranishi charts. Note also that compatible coordinate charts pull back to compatible coordinate charts under base changes of $\hat{\mathbf{B}}$.

Definition 2.27. *A Kuranishi chart $(\mathcal{U}, V, \hat{f}/G)$ is extendible if it has an extension, which is a Kuranishi chart $(\mathcal{U}^{\sharp}, V^{\sharp}, \hat{f}^{\sharp}/G)$ so that*

- *there exists a continuous map*

$$\rho : \mathcal{U}^{\sharp} \longrightarrow (0, 1]$$

so that

—

$$\mathcal{U} = \rho^{-1}((1/2, 1])$$

- *for any $t > 0$, all holomorphic curves in the closure of $\rho^{-1}((t, 1])$ within \mathcal{M}^{st} are contained within $\rho^{-1}([t, 1])$,*
- *for any $t > 0$, the closure within \mathcal{M}^{st} of the subset of \hat{f}^{\sharp} where $\rho > t$ is contained within \hat{f}^{\sharp} .*
- *V is the restriction of V^{\sharp} to \mathcal{U} , and \hat{f} is the restriction of \hat{f}^{\sharp} to \mathcal{U} .*

Note that extendible Kuranishi charts pull back to extendible Kuranishi charts. The condition of extendability is designed to prevent pathological behavior from occurring at the boundary of Kuranishi charts. We shall have cause to repeatedly shrink the size of extensions during inductive constructions—restricting $(\mathcal{U}^{\sharp}, V^{\sharp}, \hat{f}^{\sharp}/G)$ to $\rho > t$ for any $t \in (0, 1/2)$ gives an extension of $(\mathcal{U}, V, \hat{f}/G)$.

As we shall have no reason to distinguish between V and V^{\sharp} , we shall sometimes use the notation V to refer to V^{\sharp} .

Definition 2.28. *A collection of extendible Kuranishi charts is locally finite if there exists an extension $(\mathcal{U}^{\sharp}, V^{\sharp}, \hat{f}^{\sharp}/G)$ of each Kuranishi chart $(\mathcal{U}, V, \hat{f}/G)$ so that each holomorphic curve has a neighborhood which intersects only finitely many of the \mathcal{U}^{\sharp} and each \mathcal{U}^{\sharp} intersects only finitely many of the other \mathcal{U}^{\sharp} .*

The collection is compatible if every pair of extended Kuranishi charts is compatible.

The collection is said to cover any substack of \mathcal{M}^{st} which is covered by $\{\mathcal{U}\}$.

Definition 2.29. *An embedded Kuranishi structure on $\mathcal{M} \subset \mathcal{M}^{st}$ is a countable, locally finite, compatible collection of extendible Kuranishi charts $\{(\mathcal{U}_i, V_i, \hat{f}_i/G_i)\}$ which covers $\mathcal{M} \subset \mathcal{M}^{st}$.*

In [11], it is proved that in many cases, the moduli stack of stable holomorphic curves, \mathcal{M} , has the following compactness property: The map $\mathcal{M} \rightarrow \mathbf{B}_0$ is proper when $\mathcal{M} \subset \mathcal{M}^{st}$ is restricted to any connected component of \mathcal{M}^{st} . If \mathcal{M} satisfies this compactness property, then Theorem 7.2 on page 69 states that there exists an embedded Kuranishi structure on \mathcal{M} , and that given any submersion $\Phi : \mathcal{M}^{st} \rightarrow \mathbf{X}$, this embedded Kuranishi structure on \mathcal{M} may be chosen so that all Kuranishi charts are Φ -submersive.

Note that given any base change of our family of targets,

$$\begin{array}{ccc} \hat{\mathbf{B}}' & \longrightarrow & \hat{\mathbf{B}} \\ \downarrow & & \downarrow \\ \mathbf{B}'_0 & \longrightarrow & \mathbf{B}_0 \end{array}$$

the pullback of any embedded Kuranishi structure on $\mathcal{M}(\hat{\mathbf{B}})$ is an embedded Kuranishi structure on $\mathcal{M}(\hat{\mathbf{B}}')$. Corollary 7.4 on page 71 proves that any two embedded Kuranishi structures on $\mathcal{M}(\hat{\mathbf{B}})$ are homotopic in the sense that there exists an embedded Kuranishi structure on $\mathcal{M}(\hat{\mathbf{B}} \times \mathbb{R})$ which pulls back to give each of the original embedded Kuranishi structures under the inclusions of $\hat{\mathbf{B}}$ over 0 and 1 in \mathbb{R} .

Given an embedded Kuranishi structure $\{(\mathcal{U}_i, V_i, \hat{f}_i/G_i)\}$, we shall construct in section 8 a canonical homotopy class of complex structure on $T_f \mathbf{F}(\hat{f}_i^\#) \downarrow_{\mathbf{B}_0}$ for all holomorphic curves f in \hat{f}_i so that

- All \mathbb{R} -nil vectors at holomorphic curves are given the canonical complex structure
- The action of G_i is complex in the sense that if an element of G_i sends f to f' , the corresponding map $T_f \mathbf{F}(\hat{f}_i^\#) \downarrow_{\mathbf{B}_0} \rightarrow T_{f'} \mathbf{F}(\hat{f}_i^\#) \downarrow_{\mathbf{B}_0}$ is complex.
- If f is a holomorphic curve in $\hat{f}_i^\#$ and $\hat{f}_j^\#$ and $V_i(f) \subset V_j(f)$, then the following is a complex short exact sequence

$$0 \rightarrow T_f \mathbf{F}(\hat{f}_i^\#) \downarrow_{\mathbf{B}_0} \rightarrow T_f \mathbf{F}(\hat{f}_j^\#) \xrightarrow{D\pi_{V_i} \bar{\partial}} V_j/V_i(f) \rightarrow 0$$

- There exists a complex structure on $T \mathbf{F}(\hat{f}_i^\#) \downarrow_{\mathbf{B}_0}$ defined on a neighborhood of the holomorphic curves with restricts to the given complex structure at holomorphic curves.

We shall also show in section 8 that given a holomorphic submersion

$$\begin{array}{ccc} \mathcal{M}^{st} & \xrightarrow{\Phi} & \mathbf{X} \\ \downarrow & & \downarrow \\ \mathbf{B}_0 & \longrightarrow & \mathbf{X}_0 \end{array}$$

we may construct this complex structure so that

$$T_f \Phi : T_f \mathbf{F}(\hat{f}_i) \downarrow_{\mathbf{B}_0} \rightarrow T_{\Phi(f)} \mathbf{X} \downarrow_{\mathbf{X}_0}$$

is complex for all f in \hat{f}_i .

3. TRIVIALIZATIONS AND PRE-OBSTRUCTION MODELS

This section summarizes the results of [13] which will be necessary for this paper. The notion from [13] of a trivialization (F, Φ) associated to a family of curves \hat{f} allows us to identify sections of $\hat{f}^*T_{\text{vert}}\hat{\mathbf{B}}$ with families of curves parametrized by $\mathbf{C}(\hat{f})$, and to identify $\bar{\partial}$ of such a family of curves with a section of $\mathcal{Y}(\hat{f})$.

Definition 3.1. *Given a $C^{\infty,1}$ family of curves \hat{f} , a choice of trivialization (F, Φ) is*

- (1) *a $C^{\infty,1}$ map*

$$\begin{array}{ccc} \hat{f}^*T_{\text{vert}}\hat{\mathbf{B}} & \xrightarrow{F} & \hat{\mathbf{B}} \\ \downarrow & & \downarrow \\ \mathbf{F} & \longrightarrow & \mathbf{B}_0 \end{array}$$

so that

- (a) *F restricted to the zero section is equal to \hat{f} ,*
 - (b) *TF restricted to the canonical inclusion of $\hat{f}^*T_{\text{vert}}\hat{\mathbf{B}}$ over the zero section is equal to the identity,*
 - (c) *TF restricted to the vertical tangent space at any point of $\hat{f}^*T_{\text{vert}}\hat{\mathbf{B}}$ is injective.*
- (2) *A $C^{\infty,1}$ isomorphism from the bundle $F^*T_{\text{vert}}\hat{\mathbf{B}}$ to the vertical tangent bundle of $\hat{f}^*T_{\text{vert}}\hat{\mathbf{B}}$ which preserves J , and which restricted to the zero section of $\hat{f}^*T_{\text{vert}}\hat{\mathbf{B}}$ is the identity.*

*In other words, if $\pi : \hat{f}^*T_{\text{vert}}\hat{\mathbf{B}} \rightarrow \mathbf{C}(\hat{f})$ denotes the vector bundle projection, a $C^{\infty,1}$ isomorphism between $F^*T_{\text{vert}}\hat{\mathbf{B}}$ and $\pi^*\hat{f}^*T_{\text{vert}}\hat{\mathbf{B}}$ which preserves the almost complex structure J on $T_{\text{vert}}\hat{\mathbf{B}}$. This can be written as a $C^{\infty,1}$ vector bundle map*

$$\begin{array}{ccc} F^*T_{\text{vert}}\hat{\mathbf{B}} & \xrightarrow{\Phi} & \hat{f}^*T_{\text{vert}}\hat{\mathbf{B}} \\ \downarrow & & \downarrow \pi \\ \hat{f}^*T_{\text{vert}}\hat{\mathbf{B}} & \xrightarrow{\pi} & \mathbf{C}(\hat{f}) \end{array}$$

*which is the identity when the vector bundle $F^*T_{\text{vert}}\hat{\mathbf{B}} \rightarrow \hat{f}^*T_{\text{vert}}\hat{\mathbf{B}}$ is restricted to the zero section of $\hat{f}^*T_{\text{vert}}\hat{\mathbf{B}}$.*

A trivialization allows us to define $\bar{\partial}$ of a section

$$\nu : \mathbf{C}(\hat{f}) \rightarrow \hat{f}^*T_{\text{vert}}\hat{\mathbf{B}}$$

as follows: $F \circ \nu$ is a map $\mathbf{C}(\hat{f}) \rightarrow \hat{\mathbf{B}}$, so $\bar{\partial}(F \circ \nu)$ is a section of

$$\mathcal{Y}(F \circ \nu) = \Gamma^{0,1} \left(T_{\text{vert}}^*\mathbf{C}(\hat{f}) \otimes (F \circ \nu)^*T_{\text{vert}}\hat{\mathbf{B}} \right)$$

Applying the map Φ to the second component of this tensor product gives an identification of $\mathcal{Y}(F \circ \nu)$ with $\mathcal{Y}(\hat{f})$, so we may consider $\bar{\partial}(F \circ \nu)$ to be a section of $\mathcal{Y}(\hat{f})$. Define $\bar{\partial}\nu$ to be this section of $\mathcal{Y}(\hat{f})$.

For example, we may construct a trivialization by extending \hat{f} to a map F satisfying the above conditions (for instance by choosing a smooth connection on $T_{\text{vert}}\hat{\mathbf{B}}$ and reparametrising the exponential map on a neighborhood of the zero

section in $f^*T_{vert}\hat{\mathbf{B}}$, and letting Φ be given by parallel transport along a linear path to the zero section using a smooth J preserving connection on $T_{vert}\hat{\mathbf{B}}$.

Given a choice of trivialization for \hat{f} and a $C^{\infty,1}$ section ν of \hat{f} , there is an induced choice of trivialization for the family $F(\nu)$, described in [13].

Definition 3.2. A $C^{\infty,1}$ pre obstruction model $(\hat{f}, V, F, \Phi, \{s_i\})$, is given by

- (1) a $C^{\infty,1}$ family of curves \hat{f}
- (2) a choice of trivialization (F, Φ) for \hat{f} in the sense of definition 3.1.
- (3) a finite collection $\{s_i\}$ of extra marked points on $\mathbf{C}(\hat{f})$ corresponding to $C^{\infty,1}$ sections

$$s_i : \mathbf{F}(\hat{f}) \longrightarrow \mathbf{C}(\hat{f})$$

so that restricted to any curve \mathbf{C} in $\mathbf{C}(\hat{f})$, these marked points are distinct and contained inside the smooth components of \mathbf{C} .

- (4) a finite dimensional sub-vectorbundle V of $\mathcal{Y}(\hat{f})$. (In other words, a locally free, finitely generated subsheaf of $\mathcal{Y}(\hat{f})$, which is a sheaf of $C^{\infty,1}(\mathbf{F}(\hat{f}))$ -modules)

We shall usually use the notation (\hat{f}, V) for a pre obstruction bundle.

Definition 3.3. Given any family of curves, \hat{f} , with a collection, $\{s_i\}$, of extra marked points on $\mathbf{C}(\hat{f})$, let $X^{\infty,1}(\hat{f})$ indicate the space of $C^{\infty,1}$ sections of $f^*T_{vert}\hat{\mathbf{B}}$ which vanish on the extra marked points $\{s_i\}$ on $\mathbf{C}(\hat{f})$.

Note that both $X^{\infty,1}(\hat{f})$ and $\mathcal{Y}(\hat{f})$ are complex vector spaces because they consist of sections of complex vector bundles.

We may restrict any pre obstruction bundle (\hat{f}, V) to a single curve f in \hat{f} . The restriction of V to this curve f is a finite dimensional linear subspace $V(f) \subset \mathcal{Y}(f)$.

Let $D\bar{\partial}(f) : X^{\infty,1}(f) \longrightarrow \mathcal{Y}(f)$ indicate the derivative of $\bar{\partial}$ at $0 \in X^{\infty,1}(f)$. We are most interested in pre obstruction bundles (\hat{f}, V) containing curves holomorphic curves f so that that $D\bar{\partial}(f)$ is injective and has image complementary to $V(f)$.

Note that if there are enough extra marked points that $\mathbf{C}(f)$ with these extra marked points is stable, $X^{\infty,1}(f)$ is a linear subspace of $T_f\mathcal{M}^{st} \downarrow_{\mathbf{B}_0} \subset T_f\mathcal{M}^{st}$. If f is also holomorphic, $X^{\infty,1}$ is a \mathbb{C} -linear subspace of $T_f\mathcal{M}^{st} \downarrow_{\mathbf{B}_0}$, and $D\bar{\partial}(f)$ corresponds to the restriction of $D\bar{\partial} : T_f\mathcal{M}^{st} \longrightarrow \mathcal{Y}(f)$ to this subspace.

To describe the importance of pre-obstruction models, we shall need the notion of a simple perturbation below.

Definition 3.4. Given a trivialization for \hat{f} , a simple perturbation of $\bar{\partial}$ is a map

$$\bar{\partial}' : X^{\infty,1}(\hat{f}) \longrightarrow \mathcal{Y}(\hat{f})$$

so that

$$\bar{\partial}'\nu = \bar{\partial}(\nu) + \Psi(\nu)$$

where Ψ is a (usually nonlinear) $C^{\infty,1}$ map

$$\begin{array}{ccc} f^*T_{vert}\hat{\mathbf{B}} & \xrightarrow{\Psi} & Y(\hat{f}) \\ & \searrow & \swarrow \\ & \mathbf{C}(\hat{f}) & \end{array}$$

which is the zero section restricted to edges of curves in $\mathbf{C}(\hat{f})$.

Example 3.5 (Construction of a simple perturbation).

Let θ be a section of $\Gamma^{0,1}\left(T_{vert}^*\mathbf{C}(\hat{f}) \otimes T_{vert}\hat{\mathbf{B}}\right)$ and suppose that \hat{f} comes with a trivialization (F, Φ) . A section ν of $\hat{f}^*T_{vert}\hat{\mathbf{B}}$ defines a map

$$(\text{id}, F(\nu)) : \mathbf{C}(\hat{f}) \longrightarrow \mathbf{C}(\hat{f}) \times \hat{\mathbf{B}}$$

Pulling back the section θ over $(\text{id}, F(\nu))$ gives a section of $\mathcal{Y}(F(\nu))$, which we can identify as a section of $\mathcal{Y}(\hat{f})$ using the map Φ from our trivialization. Therefore, we get a modification $\bar{\partial}'$ of the usual $\bar{\partial}$ equation on sections $\hat{f}^*T_{vert}\hat{\mathbf{B}}$ given by the trivialization

$$\bar{\partial}'\nu := \bar{\partial}\nu - \Phi((\text{id}, F(\nu))^*\theta)$$

As F , θ and Φ are $C^{\infty,1}$ maps, we may define a $C^{\infty,1}$ map $\psi(\nu) := -\Phi((\text{id}, F(\nu))^*\theta)$, so $\bar{\partial}'$ is a simple perturbation of $\bar{\partial}$.

The following theorem is the main theorem of [13].

Theorem 3.6. *Suppose that (\hat{f}, V) is a $C^{\infty,1}$ pre obstruction model containing the curve f , so that $\bar{\partial}f \in V(f)$, and*

$$D\bar{\partial}(f) : X^{\infty,1}(f) \longrightarrow \mathcal{Y}(f)$$

is injective and has image complementary to $V(f)$.

Then the restriction (\hat{f}', V) of (\hat{f}, V) to some open neighborhood of f satisfies the following:

There exists a neighborhood U of $\bar{\partial}$ in the space of $C^{\infty,1}$ perturbations of $\bar{\partial}$ and a neighborhood O of 0 in $X^{\infty,1}(\hat{f}')$ so that

- (1) *Given any curve f' in \hat{f}' , section $\nu \in O$, and simple perturbation $\bar{\partial}'$ of $\bar{\partial}$ in U ,*

$$D\bar{\partial}'(\nu(f')) : X^{\infty,1}(\nu(f')) \longrightarrow \mathcal{Y}(\nu(f'))$$

is injective and has image complementary to $V(f')$.

- (2) *For any $\bar{\partial}' \in U$, there exists some $\nu \in O$ and a section v of V so that*

$$\bar{\partial}'\nu = v$$

The sections ν and v are unique in the following sense: Given any curve g in \hat{f}' , let $\nu(g)$ and $O(g)$ be the relevant restrictions of ν and O to g . Then $\nu(g)$ is the unique element of $O(g)$ so that $\bar{\partial}'\nu(g) \in V(g)$.

The map $U \longrightarrow O$ which sends $\bar{\partial}'$ to the corresponding solution ν is continuous in the $C^{\infty,1}$ topologies on U and O .

The same theorem also holds with any simple perturbation $\bar{\partial}'$ of $\bar{\partial}$ used in place of $\bar{\partial}$.

We shall also need the following corollary of Theorem 3.6.

Corollary 3.7. *Let (\hat{f}, V) be a $C^{\infty,1}$ pre-obstruction model containing f so that $\bar{\partial}'f$ is tangent to V at f , and $D\bar{\partial}'(f)$ is transverse to V . Then the unique section ν of $X^{\infty,1}(\hat{f})$ so that $\bar{\partial}'(\nu)$ is a section of V is tangent to 0 at f .*

Proof: We may restrict (\hat{f}, V) to a pre obstruction model (\hat{f}', V) where \hat{f}' is parametrized by \mathbb{R} and f is the curve over 0. We may also restrict $\bar{\partial}'$ to (\hat{f}', V) . The uniqueness part of Theorem 3.6 implies that ν pulls back to the section ν' so that $\bar{\partial}'\nu'$ is a section of V . It therefore suffices to prove that ν' is tangent to the zero section at 0.

Claim 2.14 implies that $\bar{\partial}\hat{f}'$ is equal to a section of V plus $t^2\theta$ where θ is a section of $\mathcal{Y}(\hat{f}')$, and t is the coordinate on \mathbb{R} . Consider the family of curves $\hat{f}' \times \mathbb{R}$. We may pull back our pre obstruction model, $\bar{\partial}'$ and θ to $\hat{f}' \times \mathbb{R}$. Theorem 3.6 then

implies that in a neighborhood of $(f, 0)$, there is a unique $C^{\infty,1}$ solution ψ to the equation

$$(\bar{\partial}'\psi - x\theta) \in V$$

where x is the coordinate on the extra \mathbb{R} factor of $\hat{f}' \times \mathbb{R}$. The uniqueness part of Theorem 3.6 implies that

$$\psi(t, t^2) = 0$$

and

$$\psi(t, 0) = \nu'(t)$$

therefore ν' is tangent to the zero section at $t = 0$, as required. \square

The following theorem, proved in [13] implies that for any simple perturbation $\bar{\partial}'$ of $\bar{\partial}$, we may treat $D\bar{\partial}'(f)$ like it is a Fredholm operator, and that we may define orientations on the kernel and cokernel of $D\bar{\partial}'(f)$ by choosing a homotopy of $D\bar{\partial}'(f)$ to a complex map.

Theorem 3.8. *Given any $C^{\infty,1}$ family of curves \hat{f} with a trivialization, a set $\{s_i\}$ of extra marked points and a simple perturbation $\bar{\partial}'$ of $\bar{\partial}$, the following is true:*

- (1) *for every curve f in \hat{f} ,*

$$D\bar{\partial}'(f) : X^{\infty,1}(f) \longrightarrow \mathcal{Y}(f)$$

is a linear map which has a closed image and finite dimensional kernel and cokernel.

- (2) *The dimension of the kernel minus the dimension of the cokernel of $D\bar{\partial}'(f)$ is a topological invariant*

$$2c_1 - 2n(g + s - 1)$$

where c_1 is the integral of the first Chern class of J over the curve f , $2n$ is the relative dimension of $\hat{\mathbf{B}} \longrightarrow \mathbf{B}_0$, g is the genus of the domain of f , and s is the number of extra marked points on which sections in $X^{\infty,1}$ must vanish.

- (3) *If $D\bar{\partial}'(f)$ is injective, then for all f' in an open neighborhood of f in \hat{f} , $D\bar{\partial}'(f')$ is injective.*
 (4) *If $D\bar{\partial}'(f)$ is injective for all f in \hat{f} , then there is a $C^{\infty,1}$ vector bundle K over $\mathbf{F}(\hat{f})$ with an identification of the fiber over f with the dual of the cokernel of $D\bar{\partial}'(f)$,*

$$K(f) = \mathcal{Y}(f) / D\bar{\partial}'(f)(X^{\infty,1}(f))$$

so that given any section θ in $\mathcal{Y}(\hat{f})$, the corresponding section of K is $C^{\infty,1}$.

The set of maps

$$X^{\infty,1}(\hat{f}) \longrightarrow \mathcal{Y}(\hat{f})$$

equal to some $D\bar{\partial}'$ for some simple perturbation $\bar{\partial}'$ of $\bar{\partial}$ is convex and contains the complex map

$$\frac{1}{2}(\nabla \cdot + J \circ \nabla \cdot \circ j) : X^{\infty,1}(\hat{f}) \longrightarrow \mathcal{Y}(\hat{f})$$

for any $C^{\infty,1}$ connection ∇ on $T_{\text{vert}}\hat{\mathbf{B}}$ which preserves J .

The set of all such $D\bar{\partial}' : X^{\infty,1}(\hat{f}) \longrightarrow \mathcal{Y}(\hat{f})$ is independent of choice of trivialization for \hat{f} .

4. EVALUATION MAPS

4.1. The map ev^0 .

In this section we show that the moduli stack of $C^{\infty,1}$ families of stable curves, $\mathcal{M}(pt)$, is an exploded orbifold, and construct a fiberwise holomorphic map

$$ev^0 : \mathcal{M}^{\infty,1}(\hat{\mathbf{B}}) \longrightarrow \mathcal{M}(pt)$$

Remark 4.1. *We shall show that $\mathcal{M}(pt)$ may be regarded the explosion of the usual Deligne-Mumford space considered as a complex orbifold with normal crossing divisors corresponding to the boundary. More specifically, the usual Deligne-Mumford space \bar{M} may be regarded as a stack over the category of complex manifolds with normal crossing divisors. It is locally represented by U/G where U is some complex manifold with normal crossing divisors, and some action of G . The inverse image U' of U under the forgetful map*

$$\bar{M}^{+1} \longrightarrow \bar{M}$$

from Deligne-Mumford space with one extra marked point is a family of curves

$$U' \longrightarrow U$$

with automorphism group G . The explosion functor applied to this family gives a family of exploded curves

$$\text{Expl } U' \longrightarrow \text{Expl } U$$

with a group G of automorphisms which locally represents the moduli stack of stable exploded curves.

Thinking of Deligne-Mumford space as an orbifold locally equivalent to (the stack of holomorphic maps into) U/G , we shall prove that $\mathcal{M}(pt)$ is locally equivalent to (the stack of all $C^{\infty,1}$ maps into) $\text{Expl } U/G$, and may be regarded as the explosion of Deligne-Mumford space.

The paper [15] introduces the concept of families of curves with universal tropical structure. We shall need the following two results from [15]:

Theorem 4.2. *For any stable curve f in $\mathcal{M}^{st}(\hat{\mathbf{B}})$ with domain not equal to \mathbf{T} , there exists a family of curves \hat{f} containing f with universal tropical structure so that*

- (1) *there is a group G of automorphisms of \hat{f} which acts freely and transitively on the set of maps of f into \hat{f} .*
- (2) *There is only one strata \mathbf{F}_0 of $\mathbf{F}(\hat{f})$ which contains the image of a map $f \longrightarrow \hat{f}$, and the smooth part of this strata, $[\mathbf{F}_0]$ is a single point.*
- (3) *The action of G on $[\mathbf{C}(\hat{f})]$ restricted to the inverse image of $[\mathbf{F}_0]$ is effective, so G may be regarded as a subgroup of the group of automorphisms of $[f]$.*

Lemma 4.3. *Let \hat{f} be a family of curves with universal tropical structure at f . Let \hat{h} be a family of curves containing a curve f' with a degree 1 holomorphic map*

$$\phi : \mathbf{C}(f') \longrightarrow \mathbf{C}(f)$$

so that $f' = f \circ \phi$. Then by restricting to a neighborhood of f' in \hat{h} there exists an extension of ϕ to a map

$$\begin{array}{ccc} \mathbf{C}(\hat{h}) & \xrightarrow{\Phi} & \mathbf{C}(\hat{f}) \\ \downarrow & & \downarrow \\ \mathbf{F}(\hat{h}) & \longrightarrow & \mathbf{F}(\hat{f}) \end{array}$$

so that in a metric on $\hat{\mathbf{B}}$, the distance between the maps \hat{h} and $\hat{f} \circ \Phi$ is bounded.

Lemma 4.4. *Suppose that \hat{f} is a family of stable curves (mapping to a point) with universal tropical structure at some curve f in \hat{f} so that the map*

$$T_f \mathbf{F}(\hat{f}) \longrightarrow T_f \mathcal{M}(pt) := T_f \mathcal{M}^{st}(pt)$$

is bijective. Then given any family of curves \hat{h} containing a curve f' with the same genus as f and with a degree 1 holomorphic map

$$\mathbf{C}(f') \longrightarrow \mathbf{C}(f)$$

there exists an open neighborhood of f' in \hat{h} with an extension of the above map to a fiberwise holomorphic map

$$\begin{array}{ccc} \mathbf{C}(\hat{h}) & \longrightarrow & \mathbf{C}(\hat{f}) \\ \downarrow & & \downarrow \\ \mathbf{F}(\hat{h}) & \longrightarrow & \mathbf{F}(\hat{f}) \end{array}$$

Proof:

By restricting to a neighborhood of f' in \hat{h} if necessary, Lemma 4.3 gives a (not necessarily holomorphic) map

$$\begin{array}{ccc} \mathbf{C}(\hat{h}) & \longrightarrow & \mathbf{C}(\hat{f}) \\ \downarrow & & \downarrow \\ \mathbf{F}(\hat{h}) & \longrightarrow & \mathbf{F}(\hat{f}) \end{array}$$

which is the given holomorphic map on $\mathbf{C}(f')$. Pull back $T\mathbf{F}(\hat{f})$ along this map to give a vector bundle E over $\mathbf{F}(\hat{h})$, then pull back \hat{h} over $E \longrightarrow \mathbf{F}(\hat{f})$ to give a family \hat{h}' of curves:

$$\begin{array}{ccc} \mathbf{C}(\hat{h}') & \longrightarrow & \mathbf{C}(\hat{h}) \\ \downarrow & & \downarrow \\ \mathbf{F}(\hat{h}') = E & \longrightarrow & \mathbf{F}(\hat{h}) \end{array}$$

Now construct a map

$$\begin{array}{ccc} \mathbf{C}(\hat{h}') & \xrightarrow{\hat{h}'} & \mathbf{C}(\hat{f}) \\ \downarrow & & \downarrow \\ \mathbf{F}(\hat{h}') & \longrightarrow & \mathbf{F}(\hat{f}) \end{array}$$

which is the previous map restricted to the zero section of $E = \mathbf{F}(\hat{h}')$, and so that the derivative along the fibers of E of the map $E \longrightarrow \mathbf{F}(\hat{f})$ is the identity when restricted to the zero section.

Note that calling the above map \hat{h}' involves a sleight of hand, because before, our notation implied that \hat{h}' was a family of curves mapping to a point instead of into $\mathbf{C}(\hat{f})$. There is a canonical inclusion of $\mathbf{C}(f')$ into $\mathbf{C}(\hat{h}')$ which corresponds to the inclusion of $\mathbf{C}(f')$ into $\mathbf{C}(\hat{h})$ followed by the inclusion of $\mathbf{C}(\hat{h})$ into $\mathbf{C}(\hat{h}')$ corresponding to the zero section of E . We shall further abuse notation by calling f' the restriction of \hat{h}' to $\mathbf{C}(f') \subset \mathbf{C}(\hat{h}')$, so f' now means a map $\mathbf{C}(f') \longrightarrow \mathbf{C}(\hat{f})$ which corresponds to the original holomorphic map $\mathbf{C}(f') \longrightarrow \mathbf{C}(\hat{f})$.

Consider $\mathbf{C}(\hat{f}) \longrightarrow \mathbf{F}(\hat{f})$ as a family of targets. We may choose a trivialization for \hat{h}' in the sense of Definition 3.1 using a connection ∇ on $T_{vert} \mathbf{C}(\hat{f})$ which is a holomorphic connection $\mathbf{C}(f)$. Consider the linearization of the $\bar{\partial}$ operator at f' in \hat{h}' using this trivialization.

$$D\bar{\partial}(f') : X^{\infty,1}(f') \longrightarrow \mathcal{Y}(f')$$

$X^{\infty,1}(f')$ in this setting is equal to $C^{\infty,1}$ sections of the complex vector bundle $(f')^* T\mathbf{C}(f)$ and $D\bar{\partial}(f')$ is equal to $\frac{1}{2}(\nabla + j \circ \nabla \circ j)$. Any component of $\mathbf{C}(f')$

which maps to a point in $[\mathbf{C}(f')]$ is a sphere, and $D\bar{\partial}(f')$ is equal to the usual $\bar{\partial}$ operator acting on complex valued functions on these components. The usual $\bar{\partial}$ operator acting on \mathbb{C} valued functions on a sphere has no cokernel and has kernel equal to the constant functions. It follows that any element of the kernel of $D\bar{\partial}(f)$ must be constant on all these components, and must therefore be the pullback of a holomorphic vector field from $T\mathbf{C}(f)$ because $\mathbf{C}(f') \rightarrow \mathbf{C}(f)$ has degree 1. As $\mathbf{C}(f)$ is stable, it has no nonzero holomorphic vector fields, so $D\bar{\partial}(f)$ has trivial kernel.

As all the strata of $\mathbf{C}(f')$ which are collapsed under the map $\mathbf{C}(f') \rightarrow \mathbf{C}(f)$ are spheres or extra edges, and each connected component of $\mathbf{C}(f')$ which is collapsed has zero genus, any section in $\mathcal{Y}(f')$ is equal to $D\bar{\partial}(f')$ of a section in $X^{\infty,1}(f')$ when restricted to the inverse image of a small enough open subset of $\mathbf{C}(f)$. It follows that the cokernel of $D\bar{\partial}(f')$ is equal to the cokernel of $\frac{1}{2}(\nabla + j \circ \nabla \circ j)$ acting on the space of vectorfields on $\mathbf{C}(f)$.

As noted in Lemma 2.11, if j indicates the almost complex structure on $\mathbf{C}(f)$ and v is a vectorfield on $\mathbf{C}(f)$,

$$\nabla v + j \circ \nabla(jv) = -j \circ L_v j$$

As $T_f \mathcal{M}^{st}(pt)$ is defined to be the quotient of $\mathcal{Y}(f)$ by the image of $L_{(\cdot)} j$, and $\frac{1}{2}(\nabla + j \circ \nabla \circ j)$ and $L_{(\cdot)} j$ are both complex linear, $T_f \mathcal{M}^{st}(pt)$ is equal to the cokernel of $\frac{1}{2}(\nabla + j \circ \nabla \circ j)$ and hence the cokernel of $D\bar{\partial}(f')$.

As $T_f \mathbf{F}(\hat{f})$ is equal to $T_f \mathcal{M}^{st}(pt)$, which is equal to the cokernel of $D\bar{\partial}(f')$, the derivative of $\bar{\partial}\hat{h}'$ at f' in the fiber direction of E is an isomorphism onto the cokernel of $D\bar{\partial}(f')$.

We may choose a pre obstruction model (\hat{h}', V) so that $V(f')$ is equal to the image of the derivative of $\bar{\partial}\hat{h}'$ in the E direction at f' . Then Theorem 3.6 implies that in a neighborhood of f' in \hat{h}' , there exists a $C^{\infty,1}$ section ν of $(\hat{h}')^* T_{vert} \mathbf{C}(\hat{f})$ so that $\bar{\partial}\nu$ is a section of V . As f' in \hat{h} started off holomorphic, and the derivative of $\bar{\partial}\hat{h}'$ in the E direction at f' is in $V(f)$, Corollary 3.7 then implies that ν vanishes to first order in the E direction at f . Therefore the derivative of $\bar{\partial}\nu$ in the E direction at f' is equal to the derivative of $\bar{\partial}\hat{h}'$ in the E direction at f' , which is an isomorphism onto $V(f)$.

Therefore, on a neighborhood of f' , the intersection of $\bar{\partial}\nu$ with the zero section is a section of $E \rightarrow \mathbf{F}(\hat{h})$. This section composed with the map $F(\nu)$ is the required $C^{\infty,1}$ holomorphic map from a neighborhood of f' in \hat{h} to \hat{f} .

□

Lemma 4.5. *Suppose that \hat{f} is a family of stable curves (mapping to a point) with a finite group G of automorphisms so that for every curve f in \hat{f}*

- (1) *G acts freely and transitively on the set of maps $f \rightarrow \hat{f}$, and each of these maps have different smooth part $[f] \rightarrow [\hat{f}]$.*
- (2) *\hat{f} has universal tropical structure at f .*
- (3) *The map*

$$T_f \mathbf{F}(\hat{f}) \rightarrow T_f \mathcal{M}(pt) := T_f \mathcal{M}^{st}(pt)$$

is bijective.

Then there is an open substack of $\mathcal{M}(pt)$ which is equivalent to the stack of $C^{\infty,1}$ maps into $\mathbf{F}(\hat{f})/G$.

Proof:

Lemma 4.4 implies that if a curve h in \hat{h} is isomorphic to a curve in \hat{f} , then there is a map from a neighborhood of h in \hat{h} into \hat{f} which extends the given isomorphism. Let \mathcal{U} indicate the substack of $\mathcal{M}(pt)$ consisting of families of curves isomorphic to some curve in \hat{f} . The above consideration implies that the curves in \hat{h} which are in \mathcal{U} form an open subset of \hat{h} . Corollary 2.8 then implies that \mathcal{U} is open.

Now let \hat{h} be a family of curves in \mathcal{U} with two maps $\phi_i : \hat{h} \rightarrow \hat{f}$. If these two maps are different at a curve h in \hat{h} , then the $\phi_i(h)$ must differ by the action of an element of G , and therefore, their smooth parts must be different. It follows that the two maps ϕ_i are equal on a closed subset of $\mathbf{F}(\hat{h})$. Similarly, ϕ_1 is equal to ϕ_2 composed with the action of a given element of G on a closed subset of $\mathbf{F}(\hat{h})$. As restricted to any curve, ϕ_1 is always equal to ϕ_2 composed with some element of G , and G is finite, it follows that the subset on which ϕ_1 is equal to ϕ_2 composed with a given element of G is both open and closed. In particular, if \hat{h} is a connected family of curves, then ϕ_1 is equal to ϕ_2 composed with some element of G .

Around every curve h in a family \hat{h} of curves in \mathcal{U} , Lemma 4.4 gives us the existence of $|G|$ maps of a neighborhood of h in \hat{h} to \hat{f} which are permuted by the action of G on \hat{f} . The above paragraph gives a kind of uniqueness for these maps so that they patch together into a unique G -fold cover \hat{h}' of \hat{h} with a G -equivariant map into \hat{f} . In other words, there exists a unique map of \hat{h} into \hat{f}/G . It follows that \mathcal{U} is equivalent to the stack of maps into $\mathbf{C}(\hat{f})/G$. \square

Given any stable curve f , one way to represent $\mathcal{M}(pt)$ near f is to use Lemma 4.2 to construct a family containing f which obeys the first two criteria of Lemma 4.5, then extend this family to also satisfy the last condition of Lemma 4.5. A second way is to apply the explosion functor to the universal curve over Deligne-Mumford space in a neighborhood of $[f]$.

Let $\pi : U' \rightarrow U$ be a family of nodal curves with marked points (in the category of complex manifolds, or schemes over \mathbb{C}) which when quotiented by its group G of automorphisms locally represents the universal curve over Deligne-Mumford space as constructed by Deligne, Mumford and Knudsen in [1] and [8], or as constructed more geometrically by Robbin and Salamon in [16], where $\pi : U' \rightarrow U$ is called a universal unfolding.

In either case, the following holds:

- (1) G acts freely and transitively on the set of inclusions of a given nodal curve into $U' \rightarrow U$.
- (2) U and U' are (or may be considered as) complex manifolds and π is a holomorphic map.
- (3) U minus the set of smooth curves is a normal crossing divisor D , and $\pi^{-1}D$ is also a normal crossing divisor, as is the union D' of $\pi^{-1}D$ with the locus of all marked points.
- (4) If a curve $\pi^{-1}(p)$ has n nodes, then around p there exist holomorphic coordinates (z_1, \dots, z_n) centered on $p = (0, \dots, 0)$, so that D is locally in the form of $z_1 \cdots z_n = 0$. At the i th node of such a $\pi^{-1}(p)$ there are local holomorphic coordinates so that π^*z_j are coordinate functions for $j \neq i$, and there are two extra coordinate functions z_i^\pm so that

$$\pi^*z_i = z_i^+ z_i^-$$

Away from nodes, all the z_i pull back to be coordinate functions. Around a marked in $\pi^{-1}(p)$, there are local coordinates $(z, \pi^*z_1, \dots, \pi^*z_m)$ so that $D' = \{z\pi^*z_1 \cdots \pi^*z_m = 0\}$.

We may therefore apply the explosion functor (described in [14]) to π :

$$\mathrm{Expl} \pi : \mathrm{Expl} U' \longrightarrow \mathrm{Expl} U$$

Lemma 4.6. *$\mathrm{Expl} \pi$ satisfies the conditions of Lemma 4.5, so $\mathrm{Expl} U/G$ is equivalent to an open substack \mathcal{U} of $\mathcal{M}(pt)$ which consists of curves isomorphic to some curve in $\mathrm{Expl} \pi$.*

The property (4) of π implies that $\mathrm{Expl} \pi$ is a family of curves. As the smooth part of $\mathrm{Expl} \pi$ is equal to π , $\mathrm{Expl} \pi$ is a family of stable curves, and the property (1) of π implies condition (1) of Lemma 4.5.

Property (4) of π implies that the tropical structure $\mathcal{P}(x)$ of $\mathrm{Expl} U$ at any point $x \in U$ is equal to $[0, \infty)^n$ where n is the number of nodes of the curve $\pi^{-1}(x)$. It also implies that at the i th internal edge e_i of $\mathrm{Expl} \pi^{-1}(x)$, the tropical structure of $\mathrm{Expl} U'$ is given by the fiber product

$$\begin{array}{ccc} \mathcal{P}(e_i) & \longrightarrow & [0, \infty)^2 \\ \downarrow & & \downarrow a+b \\ \mathcal{P}(x) & \longrightarrow & [0, \infty) \end{array}$$

where the bottom arrow is projection onto the i th factor of $\mathcal{P}(x) = [0, \infty)^n$. It follows from Remark 3.3 of [15] that the tropical structure $\mathrm{Expl} \pi$ restricted to any curve f is the universal extension of the tropical structure of f . Therefore, condition (2) of Lemma 4.5 holds.

All that remains is to show that condition (3) of Lemma 4.5 holds. In particular, we must show that for any curve f in $\mathrm{Expl} \pi$, the map

$$T_f \mathrm{Expl} U \longrightarrow T_f \mathcal{M}(pt) := T_f \mathcal{M}^{st}(pt)$$

is bijective.

Recall from section 2.5 that $T_f \mathcal{M}(pt)$ is defined as the cokernel of the map

$$\begin{array}{ccc} \Gamma(T\mathbf{C}(f)) & \longrightarrow & \Gamma^{(0,1)}(T^*\mathbf{C}(f) \otimes T\mathbf{C}(f)) \\ v & \mapsto & L_v j \end{array}$$

As noted in Claim 2.11, we may also consider $T_f \mathcal{M}(pt)$ as equal to the cokernel of the $\bar{\partial}$ operator acting on $C^{\infty,1}$ vectorfields on $\mathbf{C}(f)$. The complex dimension of $T_f \mathcal{M}(pt)$ may be computed as $3g - 3 + k$ from Theorem 3.8 part 2 and a calculation of the first Chern class of $T\mathbf{C}(f)$ as $2 - 2g - k$ where k is the number of infinite ends of $\mathbf{C}(f)$ and g is the genus of $\mathbf{C}(f)$. As this agrees with the dimension of Deligne-Mumford space, and therefore the dimension of $\mathrm{Expl} U$, we need only check that the map $T_f \mathrm{Expl} U \longrightarrow T_f \mathcal{M}(pt)$ is injective.

Let v be in the kernel of the map $T_f \mathrm{Expl} U \longrightarrow T_f \mathcal{M}(pt)$. The description of this map from section 2.5.4 implies that there must be a lift v' of v to a section of $T \mathrm{Expl} U'$ restricted to $\mathbf{C}(f)$ so that $L_{v'} j = 0$. It remains to show that such a v' must be 0.

Use $[v']$ to indicate the image of v' under the derivative of the smooth part map $\mathrm{Expl} U' \longrightarrow U'$, and $[v]$ to indicate the image of v under the derivative of the map $\mathrm{Expl} U \longrightarrow U$. We have that $[v']$ is a lift of $[v]$ and that $L_{[v']} j = 0$ where j now indicates the fiberwise almost complex structure on $U' \longrightarrow U$ induced by the complex structure on U' . As defined, it is not obvious that $[v']$ is smooth at nodes and marked points of $[\mathbf{C}(f)]$, however, in the coordinates around nodes from property 4, such a vector field must be a constant vectorfield plus a vectorfield which is continuous and holomorphic away from the node or marked point, so it

must be smooth. As $U' \rightarrow U$ represents Deligne-Mumford space, it follows that $[v']$ and $[v]$ must be 0.

We now have that $[v'] = 0$. If $[C(f)]$ has no nodes, then around f the smooth part map $\text{Expl}U \rightarrow U$ is an isomorphism, so $[v] = 0$ implies that $v = 0$. Now suppose that $[C(f)]$ has n nodes, and use the coordinates from property (4) of Deligne-Mumford space. In particular, around the i th node we have coordinates which include z_i^\pm so that $\pi^* z_i = z_i^+ z_i^-$. These coordinates correspond to coordinates \tilde{z}_i^\pm on $\text{Expl}U'$ and a coordinate $\tilde{z}_i = \tilde{z}_i^+ \tilde{z}_i^-$ from $\text{Expl}U$. As z_i^\pm are coordinates around the i th node, and $[v' \tilde{z}_i^\pm] = [v'] z_i^\pm$, it follows that $v' \tilde{z}_i^\pm = 0$, and therefore $v \tilde{z}_i = 0$. As the divisor D in U is defined by the product of these z_i corresponding to each node, it follows that v must be equal to 0. \square

Corollary 4.7. $\mathcal{M}(pt)$ is a $C^{\infty,1}$ orbifold, and the explosion of Deligne-Mumford space.

Lemma 4.8. Suppose that \hat{f} is a connected $C^{\infty,1}$ family of curves for which $2g+n \geq 3$ where g is the genus and n is the number of external edges of curves in \hat{f} . Then there exists a unique stabilization \hat{f}^{st} of \hat{f} , which is a $C^{\infty,1}$ family of stable curves \hat{f}^{st} in $\mathcal{M}(pt)$ with a degree 1, fiberwise holomorphic, $C^{\infty,1}$ map

$$\begin{array}{ccc} \mathbf{C}(\hat{f}) & \longrightarrow & \mathbf{C}(\hat{f}^{st}) \\ \downarrow & & \downarrow \\ \mathbf{F}(\hat{f}) & \xrightarrow{\text{id}} & \mathbf{F}(\hat{f}^{st}) = \mathbf{F}(\hat{f}) \end{array}$$

which preserves the genus and number of external edges of fibers.

This defines a fiberwise holomorphic map

$$\begin{array}{ccc} \mathcal{M}^{\infty,1}(\hat{\mathbf{B}})^{+1} & \longrightarrow & \mathcal{M}(pt)^{+1} \\ \downarrow & & \downarrow \\ \mathcal{M}^{\infty,1}(\hat{\mathbf{B}}) & \xrightarrow{ev^0} & \mathcal{M}(pt) \end{array}$$

defined on the components of $\mathcal{M}^{\infty,1}(\hat{\mathbf{B}})$ for which $2g+n \geq 3$.

Proof:

We shall first consider the stabilization of a single curve f . The idea is to ‘remove’ all unstable components using a series of maps of the following two types:

- (1) If a smooth component of $\mathbf{C}(f)$ is a sphere attached to only one edge, put holomorphic coordinates on a neighborhood of the edge modeled on an open subset of $\mathbf{T}_{[0,l]}^1$ with coordinate \tilde{z} so that $[\tilde{z}]$ gives coordinates on the smooth component of $\mathbf{C}(f)$ attached to the other end of the edge. Replace this coordinate chart with the corresponding open subset of \mathbb{C} with coordinate $z = [\tilde{z}]$. There is an obvious degree one holomorphic map from our old curve to this new one that is given in this coordinate chart by $\tilde{z} \mapsto [\tilde{z}]$, and sends our unstable sphere and the edge attached to it to the point p where $z(p) = 0$. (This map is the identity everywhere else.)

As stable curves have no strata which are once-punctured spheres, the resulting holomorphic map $\mathbf{C}(f) \rightarrow \mathbf{C}'$ has the property that any holomorphic map from $\mathbf{C}(f)$ to a stable holomorphic curve factors through it.

- (2) If a smooth component of $\mathbf{C}(f)$ is a sphere attached to exactly two edges, there exists a holomorphic identification of a neighborhood of this smooth component with a refinement of an open subset of $\mathbf{T}_{[0,l]}^1$ or $\mathbf{T}_{[0,\infty)}^1$. Replace this open set with the corresponding open subset of $\mathbf{T}_{[0,l]}^1$ or $\mathbf{T}_{[0,\infty)}^1$. The degree one holomorphic map from the old exploded curve to the new one is this refinement map. (Refer to [14] for the definition of refinements.)

Again, the resulting holomorphic map $\mathbf{C}(f) \rightarrow \mathbf{C}'$ has the property that any holomorphic map of $\mathbf{C}(f)$ to a stable holomorphic curve must factor through it.

Each of the above types of maps removes one smooth component, so after applying maps of the above type a finite number of times, we obtain a connected exploded curve $\mathbf{C}(f^{st})$ with no smooth components which are spheres with one or two punctures. Our theorem's hypotheses then imply that the resulting exploded curve f^{st} must be stable.

$\mathbf{C}(f^{st})$ has the same genus and number of punctures as $\mathbf{C}(f)$, as each of the above two types of maps preserves the genus and number of punctures. The map $\mathbf{C}(f) \rightarrow \mathbf{C}(f^{st})$ is degree 1, holomorphic, and has the property that any holomorphic map from $\mathbf{C}(f)$ to a stable curve must factor through it. Any holomorphic, degree 1, genus and number of punctures preserving map between stable curves must be an isomorphism, so this universal property of $\mathbf{C}(f^{st})$ implies that f^{st} is the unique such stabilization of f .

Now consider a family \hat{f} of curves containing f , and let \hat{g} be a family of stable curves containing f^{st} , and satisfying the requirements of Lemma 4.5. (Such a family of curves exists, as proved by Lemma 4.6.) By restricting \hat{f} to a smaller neighborhood of f if necessary, Lemma 4.4 then gives us a fiberwise holomorphic map

$$\begin{array}{ccc} \mathbf{C}(\hat{f}) & \longrightarrow & \mathbf{C}(\hat{g}) \\ \downarrow & & \downarrow \\ \mathbf{F}(\hat{f}) & \longrightarrow & \mathbf{F}(\hat{g}) \end{array}$$

which extends the given holomorphic map $\mathbf{C}(f) \rightarrow \mathbf{C}(f^{st}) \subset \mathbf{C}(\hat{g})$. So long as \hat{f} is connected, this map is fiberwise degree 1 and preserves genus and number of punctures. We may therefore pull back \hat{g} over the map $\mathbf{F}(\hat{f}) \rightarrow \mathbf{F}(\hat{g})$ to obtain a stabilization of f :

$$\begin{array}{ccccc} \mathbf{C}(\hat{f}) & \longrightarrow & \mathbf{C}(\hat{f}^{st}) & \longrightarrow & \mathbf{C}(\hat{g}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{F}(\hat{f}) & \xrightarrow{id} & \mathbf{F}(\hat{f}) & \longrightarrow & \mathbf{F}(\hat{g}) \end{array}$$

Suppose that \hat{h} is a family with a stabilization \hat{h}^{st} and a map $\hat{h} \rightarrow \hat{f}$. We have assumed that there is a group G of automorphisms of \hat{g} so that given any curve h in \hat{h} , G acts freely and transitively on the set of maps $h^{st} \rightarrow \hat{g}$, and each of these maps has a different smooth part. As every fiberwise degree 1 holomorphic map $\mathbf{C}(h) \rightarrow \mathbf{C}(\hat{g})$ factors uniquely through a map $h^{st} \rightarrow \hat{g}$, it follows that G acts freely and transitively on the set of fiberwise degree 1 holomorphic maps $\mathbf{C}(h) \rightarrow \mathbf{C}(\hat{g})$, and each of these maps has a different smooth part. Following the argument of Lemma 4.5 then gives that any two fiberwise degree 1 holomorphic maps $\mathbf{C}(\hat{h}) \rightarrow \mathbf{C}(\hat{g})$ are equal on a set which is both open and closed.

Lemma 4.5 implies that if \hat{h} is small enough, there are $|G|$ $C^{\infty,1}$ maps $\hat{h}^{st} \rightarrow \hat{g}$ which are permuted by the action of G . The above argument implies that there must be one of these maps so that the following diagram commutes

$$\begin{array}{ccc} \mathbf{C}(\hat{h}) & \longrightarrow & \mathbf{C}(\hat{f}) \\ \downarrow & & \downarrow \\ \mathbf{C}(\hat{h}^{st}) & \longrightarrow & \mathbf{C}(\hat{g}) \end{array}$$

and therefore, the following diagram of $C^{\infty,1}$ maps must commute

$$\begin{array}{ccc} \mathbf{C}(\hat{h}^{st}) & \longrightarrow & \mathbf{C}(\hat{g}) \\ \downarrow & & \downarrow \\ \mathbf{F}(\hat{h}) = \mathbf{F}(\hat{h}^{st}) & & \mathbf{F}(\hat{g}) \\ \downarrow & & \downarrow \\ \mathbf{F}(\hat{f}) & \longrightarrow & \mathbf{F}(\hat{g}) \end{array}$$

so the definition of $\mathbf{C}(\hat{f}^{st})$ as a fiber product implies that the map $\hat{h}^{st} \rightarrow \hat{g}$ factors as the composition $\hat{h}^{st} \rightarrow \hat{f}^{st} \rightarrow \hat{g}$ of $C^{\infty,1}$ maps so that the following diagram commutes

$$\begin{array}{ccc} \mathbf{C}(\hat{h}) & \longrightarrow & \mathbf{C}(\hat{f}) \\ \downarrow & & \downarrow \\ \mathbf{C}(\hat{h}^{st}) & \longrightarrow & \mathbf{C}(\hat{f}^{st}) \end{array}$$

The unique factorization property of stabilizations of individual curves implies that the map $\hat{h}^{st} \rightarrow \hat{f}^{st}$ is the unique map so that the above diagram commutes. In this argument, we assumed that \hat{h} was ‘small enough’, so we have only constructed the map $\hat{h}^{st} \rightarrow \hat{f}^{st}$ locally, however, the uniqueness of this map implies that all such local constructions patch together to a globally defined map $\hat{h}^{st} \rightarrow \hat{f}^{st}$.

To summarize, we have shown that every ‘small enough’ family \hat{f} has a stabilization \hat{f}^{st} , and that if \hat{h} also has a stabilization \hat{h}^{st} and there is a map $\hat{h} \rightarrow \hat{f}$, then there is a unique map $\hat{h}^{st} \rightarrow \hat{f}^{st}$ so that the above diagram commutes. This uniqueness of locally defined stabilizations implies that locally defined stabilizations glue together. Therefore, every family of curves \hat{f} satisfying the requirements of our lemma has a stabilization, and given any map $\hat{h} \rightarrow \hat{f}$, there exists a unique map $\hat{h}^{st} \rightarrow \hat{f}^{st}$ so that the above diagram commutes.

A choice of stabilization for every family in $\mathcal{M}^{\infty,1}(\hat{\mathbf{B}})$ defines our fiberwise holomorphic map (on the connected components of $\mathcal{M}^{\infty,1}(\hat{\mathbf{B}})$ for which $2g + n \geq 3$).

$$\begin{array}{ccc} \mathcal{M}^{\infty,1}(\hat{\mathbf{B}})^{+1} & \longrightarrow & \mathcal{M}(pt)^{+1} \\ \downarrow & & \downarrow \\ \mathcal{M}^{\infty,1}(\hat{\mathbf{B}}) & \xrightarrow{ev^0} & \mathcal{M}(pt) \end{array}$$

□

4.2. The evaluation maps ev^{+n} and adding extra marked points to families.

In what follows, we define an ‘evaluation map’ for a family of curves using a functorial construction of a family of curves \hat{f}^{+n} with n extra punctures from a given family of curves \hat{f} .

Definition 4.9. *Given a submersion $f : \mathbf{D} \rightarrow \mathbf{E}$, use the following notation for the fiber product of \mathbf{D} over \mathbf{E} with itself n times:*

$$\mathbf{D}_{\mathbf{E}}^n := \mathbf{D} \times_f \mathbf{D} \times_f \mathbf{D} \times_f \cdots \times_f \mathbf{D}$$

Definition 4.10. Given a family of curves \hat{f} in $\hat{\mathbf{B}} \rightarrow \mathbf{B}_0$, define the family of curves \hat{f}^{+1} to be a family of curves in $\hat{\mathbf{B}} \times_{\mathbf{B}_0} \hat{\mathbf{B}}$ with one extra puncture

$$\begin{array}{ccc} \mathbf{C}(\hat{f}^{+1}) & \xrightarrow{\hat{f}^{+1}} & \hat{\mathbf{B}} \times_{\mathbf{B}_0} \hat{\mathbf{B}} \\ \downarrow \pi_{\mathbf{F}(\hat{f}^{+1})} & & \downarrow \\ \mathbf{F}(\hat{f}^{+1}) = \mathbf{C}(\hat{f}) & \xrightarrow{\hat{f}} & \hat{\mathbf{B}} \end{array}$$

satisfying the following conditions

- (1) The fiber of $\pi_{\mathbf{F}(\hat{f}^{+1})} : \mathbf{C}(\hat{f}^{+1}) \rightarrow \mathbf{F}(\hat{f}^{+1})$ over a point $p \in \mathbf{F}(\hat{f}^{+1}) = \mathbf{C}(\hat{f})$ is equal to the fiber of $\pi_{\mathbf{F}(\hat{f})} : \mathbf{C}(\hat{f}) \rightarrow \mathbf{F}(\hat{f})$ containing p with an extra puncture at the point p .
- (2) There exists a fiberwise holomorphic, degree 1 map

$$\begin{array}{ccc} \mathbf{C}(\hat{f}^{+1}) & \longrightarrow & \mathbf{C}(\hat{f}) \times_{\mathbf{F}(\hat{f})} \mathbf{C}(\hat{f}) \\ \downarrow & & \downarrow \\ \mathbf{C}(\hat{f}) & \xrightarrow{\text{id}} & \mathbf{C}(\hat{f}) \end{array}$$

so that the following diagram commutes.

$$\begin{array}{ccccc} & \mathbf{C}(\hat{f}^{+1}) & & & \\ & \downarrow & \searrow \hat{f}^{+1} & & \\ \mathbf{C}(\hat{f}) \times_{\mathbf{F}(\hat{f})} \mathbf{C}(\hat{f}) & \xrightarrow{\hat{f} \times \hat{f}} & \hat{\mathbf{B}} \times_{\mathbf{B}_0} \hat{\mathbf{B}} & & \\ \downarrow & & \downarrow & & \\ \mathbf{C}(\hat{f}) & \xrightarrow{\hat{f}} & \hat{\mathbf{B}} & & \\ \downarrow & & \downarrow & & \\ \mathbf{F} & \longrightarrow & \mathbf{B}_0 & & \end{array}$$

Define \hat{f}^{+0} to be \hat{f} , and for positive integers n , define \hat{f}^{+n} inductively using

$$\hat{f}^{+n} = \left(\hat{f}^{+(n-1)} \right)^{+1}$$

so \hat{f}^{+n} is a family of curves in $\hat{\mathbf{B}}_{\mathbf{B}_0}^{n+1}$.

$$\begin{array}{ccc} \mathbf{C}(\hat{f}^{+n}) & \xrightarrow{\hat{f}^{+n}} & \hat{\mathbf{B}}_{\mathbf{B}_0}^{n+1} \\ \downarrow & & \downarrow \\ \mathbf{C}(\hat{f}^{+(n-1)}) & \xrightarrow{\hat{f}^{+(n-1)}} & \hat{\mathbf{B}}_{\mathbf{B}_0}^n \\ \vdots & & \vdots \\ \mathbf{C}(\hat{f}) & \xrightarrow{\hat{f}^{+0}} & \hat{\mathbf{B}} \\ \downarrow & & \downarrow \\ \mathbf{F}(\hat{f}) & \longrightarrow & \mathbf{B}_0 \end{array}$$

Combining $\hat{f}^{+(n-1)}$ with the map $ev^0 : \mathbf{F}(\hat{f}^{+n}) \rightarrow \mathcal{M}(pt)$ given by Lemma 4.8 when n is large enough, we get the evaluation map

$$(3) \quad ev^{+n}(\hat{f}) := (ev^0, \hat{f}^{+(n-1)}) : \mathbf{F}(\hat{f}^{+n}) \rightarrow \mathcal{M}(pt) \times \hat{\mathbf{B}}_{\mathbf{B}_0}^n$$

We shall show below that given any family \hat{f} of curves, there exists a family \hat{f}^{+1} satisfying the above requirements. Then we shall show that such a family is unique and that the construction is functorial.

The total space of the domain, $\mathbf{C}(\hat{f}^{+1})$ is constructed by ‘exploding’ the diagonal of $\mathbf{C}(\hat{f}) \times_{\mathbf{F}(\hat{f})} \mathbf{C}(\hat{f})$ as follows:

Consider the diagonal map $\Delta : \mathbf{C}(\hat{f}) \rightarrow \mathbf{C}(\hat{f}) \times_{\mathbf{F}(\hat{f})} \mathbf{C}(\hat{f})$. The image of the tropical part of this map, $\underline{\Delta}$, defines a subdivision of the tropical part of $\mathbf{C}(\hat{f}) \times_{\mathbf{F}(\hat{f})} \mathbf{C}(\hat{f})$. As noted in section 10 of [14], any such subdivision determines a unique refinement $\mathbf{C}' \rightarrow \mathbf{C}(\hat{f}) \times_{\mathbf{F}(\hat{f})} \mathbf{C}(\hat{f})$. Note that the diagonal map to this refinement \mathbf{C}' is still defined,

$$\begin{array}{ccc} & & \mathbf{C}' \\ & \nearrow \Delta' & \downarrow \\ \mathbf{C}(\hat{f}) & \xrightarrow{\Delta} & \mathbf{C}(\hat{f}) \times_{\mathbf{F}(\hat{f})} \mathbf{C}(\hat{f}) \end{array}$$

and a neighborhood of the image of Δ' in \mathbf{C}' is equal to a neighborhood of 0 in a \mathbb{C} -bundle over $\mathbf{C}(\hat{f})$.

Now ‘explode’ the image of the diagonal Δ' in \mathbf{C}' to make $\mathbf{C}(\hat{f}^{+1}) \rightarrow \mathbf{C}'$ as follows: We may choose coordinate charts on \mathbf{C}' so that any coordinate chart intersecting the image of the diagonal is equal to some subset of $\mathbb{C} \times U$ where U is a coordinate chart on $\mathbf{C}(\hat{f})$, the projection to $\mathbf{C}(\hat{f})$ is the obvious projection to U , the complex structure on the fibers of this projection is equal to the standard complex structure on \mathbb{C} , and the image of the diagonal is $0 \times U$. Replace these charts with the corresponding subsets of $\mathbf{T}_1^1 \times U$, and leave coordinate charts that do not intersect the image of the diagonal unchanged. Any transition map between coordinate charts of the above type is of the form $(z, u) \mapsto (g(z, u)z, \phi(u))$ where $g(z, u)$ is \mathbb{C}^* valued and fiberwise holomorphic in z . In the corresponding ‘exploded’ charts, the corresponding transition map is given by $(\tilde{z}, u) \mapsto (g(\lceil \tilde{z} \rceil, u)\tilde{z}, \phi(u))$. The transition maps between other charts can remain unchanged. This defines $\mathbf{C}(\hat{f}^{+1})$. The map $\mathbf{C}(\hat{f}^{+1}) \rightarrow \mathbf{C}'$ is given in the above coordinate charts by $(\tilde{z}, u) \mapsto (\lceil \tilde{z} \rceil, u)$. Composing this with the refinement map $\mathbf{C}' \rightarrow \mathbf{C}(\hat{f}) \times_{\mathbf{F}(\hat{f})} \mathbf{C}(\hat{f})$ then gives a degree one fiberwise holomorphic map

$$\begin{array}{ccc} \mathbf{C}(\hat{f}^{+1}) & \rightarrow & \mathbf{C}(\hat{f}) \times_{\mathbf{F}(\hat{f})} \mathbf{C}(\hat{f}) \\ \downarrow & & \downarrow \\ \mathbf{C}(\hat{f}) & \xrightarrow{\text{id}} & \mathbf{C}(\hat{f}) \end{array}$$

The map $\hat{f}^{+1} : \mathbf{C}(\hat{f}^{+1}) \rightarrow \mathbf{C}(\hat{f}) \times_{\mathbf{F}(\hat{f})} \mathbf{C}(\hat{f})$ is given by the above constructed map $\mathbf{C}(\hat{f}^{+1}) \rightarrow \mathbf{C}(\hat{f}) \times_{\mathbf{F}(\hat{f})} \mathbf{C}(\hat{f})$ composed with the map

$$\mathbf{C}(\hat{f}) \times_{\mathbf{F}(\hat{f})} \mathbf{C}(\hat{f}) \rightarrow \hat{\mathbf{B}} \times_{\mathbf{B}_0} \hat{\mathbf{B}}$$

which is \hat{f} in each component. All the above maps are smooth or $C^{\infty,1}$ if \hat{f} is. This constructed family of curves \hat{f}^{+1} obeys the requirements of Definition 4.10.

The following lemma implies that \hat{f}^{+n} is unique (up to unique isomorphism) and that the construction of \hat{f}^{+n} is functorial.

Lemma 4.11. *Given a map of families $\hat{f} \rightarrow \hat{g}$ and families \hat{f}^{+1} and \hat{g}^{+1} satisfying the requirements of Definition 4.10 there is a unique induced map $\hat{f}^{+1} \rightarrow \hat{g}^{+1}$ so*

that the diagram

$$\begin{array}{ccc}
\mathbf{C}(\hat{f}^{+1}) & \longrightarrow & \mathbf{C}(\hat{g}^{+1}) \\
\downarrow & & \downarrow \\
\mathbf{C}(\hat{f}) \times_{\mathbf{F}(\hat{f})} \mathbf{C}(\hat{f}) & \longrightarrow & \mathbf{C}(\hat{g}) \times_{\mathbf{F}(\hat{g})} \mathbf{C}(\hat{g}) \\
\downarrow & & \downarrow \\
\mathbf{C}(\hat{f}) & \longrightarrow & \mathbf{C}(\hat{g})
\end{array}$$

commutes.

Proof:

Both $\mathbf{C}(\hat{g}^{+1})$ and $\mathbf{C}(\hat{g}) \times_{\mathbf{F}(\hat{g})} \mathbf{C}(\hat{g})$ are families over $\mathbf{C}(\hat{g})$, but the fiber in $\mathbf{C}(\hat{g}^{+1})$ over a point $p \in \mathbf{C}(\hat{g})$ has one extra puncture. Away from this puncture, the map $\mathbf{C}(\hat{g}^{+1}) \rightarrow \mathbf{C}(\hat{g}) \times_{\mathbf{F}(\hat{g})} \mathbf{C}(\hat{g})$ must be a fiberwise holomorphic isomorphism. Therefore, away from the extra puncture, the required map

$$\begin{array}{ccc}
\mathbf{C}(\hat{f}^{+1}) & \longrightarrow & \mathbf{C}(\hat{g}^{+1}) \\
\downarrow & & \downarrow \\
\mathbf{C}(\hat{f}) \times_{\mathbf{F}(\hat{f})} \mathbf{C}(\hat{f}) & \longrightarrow & \mathbf{C}(\hat{g}) \times_{\mathbf{F}(\hat{g})} \mathbf{C}(\hat{g})
\end{array}$$

exists, is unique, and is fiberwise holomorphic.

On a neighborhood of the extra puncture in the fiber over $p \in \mathbf{C}(\hat{g})$, there exists a fiberwise holomorphic exploded coordinate function \tilde{z} so that the extra puncture is at $[\tilde{z}] = 0$. The fiberwise holomorphic function $[\tilde{z}]$ is a fiberwise holomorphic coordinate function on $\mathbf{C}(\hat{g}) \times_{\mathbf{F}(\hat{g})} \mathbf{C}(\hat{g})$ which vanishes on the image of the diagonal. Therefore, $[\tilde{z}]$ pulls back to a fiberwise holomorphic coordinate function on $\mathbf{C}(\hat{f}) \times_{\mathbf{F}(\hat{f})} \mathbf{C}(\hat{f})$ which vanishes on the image of the diagonal. It follows that if \tilde{z}' is a locally defined fiberwise holomorphic coordinate function on $\mathbf{C}(\hat{f}^{+1})$ so that the extra puncture is at $[\tilde{z}'] = 0$, then the pullback of $[\tilde{z}]$ is equal to $h[\tilde{z}']$ where h is some non-vanishing, fiberwise holomorphic function. Therefore there locally exists a unique map

$$\begin{array}{ccc}
\mathbf{C}(\hat{f}^{+1}) & \longrightarrow & \mathbf{C}(\hat{g}^{+1}) \\
\downarrow & & \downarrow \\
\mathbf{C}(\hat{f}) \times_{\mathbf{F}(\hat{f})} \mathbf{C}(\hat{f}) & \longrightarrow & \mathbf{C}(\hat{g}) \times_{\mathbf{F}(\hat{g})} \mathbf{C}(\hat{g})
\end{array}$$

which is fiberwise holomorphic and pulls back \tilde{z} to $h\tilde{z}'$. As the locally defined maps we have defined all satisfy the same uniqueness property, they glue together to give the required unique map. Restricted to each fiber, this map is a holomorphic isomorphism. The fact that \hat{f}^{+1} factors through $\mathbf{C}(\hat{f}) \times_{\mathbf{F}(\hat{f})} \mathbf{C}(\hat{f})$ implies that our map $\mathbf{C}(\hat{f}^{+1}) \rightarrow \mathbf{C}(\hat{g}^{+1})$ corresponds to a unique map $\hat{f}^{+1} \rightarrow \hat{g}^{+1}$. \square

5. CORE FAMILIES

The following notion of a core family gives a way of locally describing the moduli stack \mathcal{M}^{st} of stable $C^{\infty,1}$ curves. A notion such as this is necessary, as the ‘space’ of stable curves in $\hat{\mathbf{B}} \rightarrow \mathbf{B}_0$ of a given regularity can not in general be locally modeled on even an orbifold version of a Banach space - this is because the domain of curves that we study are not fixed, and because of phenomena which would be

called bubble and node formation in the setting of smooth manifolds. (The moduli stack of stable curves could be described as a ‘orbifold’ by using an adaption to the exploded setting of the theory of polyfolds being developed by Hofer, Wysocki and Zehnder in a series of papers including [5]. An adaption of the theory of polyfolds to the exploded setting is a worthwhile direction for further research which is not explored in this paper.)

Definition 5.1. A core family of curves, $(\hat{f}/G, \{s_i\})$ for an open substack \mathcal{O} of $\mathcal{M}^{\infty,1}(\hat{\mathbf{B}})$ is:

- a $C^{\infty,1}$ family \hat{f} of stable curves with a group G of automorphisms,

$$\begin{array}{ccc} \mathbf{C}(\hat{f}) & \xrightarrow{\hat{f}} & \hat{\mathbf{B}} \\ \downarrow & & \downarrow \\ \mathbf{F}(\hat{f}) & \longrightarrow & \mathbf{B}_0 \end{array}$$

- a finite, G -invariant collection of $C^{\infty,1}$ ‘marked point’ sections $s_i : \mathbf{F}(\hat{f}) \rightarrow \mathbf{C}(\hat{f})$ which do not intersect each other, and which do not intersect the edges of the curves in $\mathbf{C}(\hat{f})$.
- A fiberwise holomorphic map

$$\begin{array}{ccc} \mathcal{O}^{+1} & \xrightarrow{\Phi} & \mathbf{C}(\hat{f})/G \\ \downarrow & & \downarrow \\ \mathcal{O} & \longrightarrow & \mathbf{F}(\hat{f})/G \end{array}$$

in the sense of definition 2.5 on page 7

so that

- (1) The map Φ applied to the family \hat{f} is the map given by quotienting $\mathbf{C}(\hat{f})$ by G .
- (2) Given any curve f in \mathcal{O} and connection on $T_{\text{vert}}\hat{\mathbf{B}}$, the maps f and $\hat{f}/G \circ \Phi$ are related by exponentiation of some vectorfield ψ which is a section of $(\hat{f}/G \circ \Phi)^*T_{\text{vert}}\hat{\mathbf{B}}$ which vanishes on the marked points corresponding to $\{s_i\}$
- (3) Conversely, given any curve f in \hat{f} and section ψ of $f^*T_{\text{vert}}\Phi$ which vanishes on marked points corresponding to $\{s_i\}$, the composition of f with exponentiation of ψ gives a new curve f' which will be in \mathcal{O} if ψ is small enough. In this case, the map $\Phi : \mathbf{C}(f') \rightarrow \mathbf{C}(\hat{f})/G$ factors through the identity map $\mathbf{C}(f') \rightarrow \mathbf{C}(f)$ and the quotient map $\mathbf{C}(f) \rightarrow \mathbf{C}(\hat{f})/G$.

Proposition 5.8 stated on page 44 constructs a core family containing any given stable holomorphic curve with at least one smooth component.

5.1. Construction of a core family.

The following theorem gives a sufficient criteria for when a given family with a collection of marked point sections is a core family:

Theorem 5.2. Let \hat{f} be a family in \mathcal{M}^{st} with a group G of automorphisms, and a finite nonempty set of disjoint sections $s_i : \mathbf{F}(\hat{f}) \rightarrow \mathbf{C}(\hat{f})$, which do not intersect the edges of the curves in $\mathbf{C}(\hat{f})$, so that the following conditions are satisfied:

- (1) For all curves f in \hat{f} , the action of G on the set of maps of f into \hat{f} is free and transitive.
- (2) For all curves f in \hat{f} , the smooth part of the domain $\mathbf{C}(f)$ with the extra marked points from $\{s_i\}$ has no automorphisms.

- (3) The action of G preserves the set of sections $\{s_i\}$, so there is some action of G as a permutation group on the set of indices $\{i\}$ so that for all $g \in G$ and s_i ,

$$s_i \circ g = g \circ s_{g(i)}$$

where the action of g is on $\mathbf{F}(\hat{f})$, $\mathbf{C}(\hat{f})$ or the set of indices $\{i\}$ as appropriate.

- (4) There exists a neighborhood U of the image of the section

$$s : \mathbf{F}(\hat{f}) \longrightarrow \mathbf{F}(\hat{f}^{+n})$$

defined by the n sections $\{s_i\}$ so that

$$ev^{+n}(\hat{f}) : \mathbf{F}(\hat{f}^{+n}) \longrightarrow \mathcal{M}(pt) \times \hat{\mathbf{B}}_{\mathbf{B}_0}^n$$

restricted to U is an equi-dimensional embedding

- (5) The tropical structure of \hat{f} is universal.
 (6) For any curve f in \hat{f} , there are exactly $|G|$ points x in $\mathbf{F}(f^{+n})$ so that $ev^{+n}(f)(x)$ is in the closure of the image of $ev^{+n}(\hat{f}) \circ s$.

Then $(\hat{f}/G, \{s_i\})$ is a core family for an open subset \mathcal{O} of $\mathcal{M}^{\infty,1}$ containing every refinement of any curve f in \hat{f} .

Proof:

Define \mathcal{O} to be the substack of $\mathcal{M}^{\infty,1}$ consisting of curves h satisfying the following properties:

- (1) $ev^{+n}(h)$ intersects closure of the image of $ev^{+n}(\hat{f}) \circ s$ exactly $|G|$ times, and each of these intersections is a transversal intersection with the image of $ev^{+n}(\hat{f}) \circ s$.
 (2) Each of the above intersection points corresponds to
- a point x in $\mathbf{F}(h^{+n})$,
 - and a curve f in \hat{f}

so that

$$ev^{+n}(h)(x) = ev^{+n}(\hat{f})(s(f))$$

The second property that h must satisfy to be in \mathcal{O} is that there exists a holomorphic refinement map

$$\Phi : \mathbf{C}(h) \longrightarrow \mathbf{C}(f)$$

so that h is equal to $\hat{f} \circ \Phi$ followed by exponentiating some vectorfield which is a section of $\hat{f} \circ \Phi^* T_{vert} \hat{\mathbf{B}}$ which vanishes at the extra n marked points on $\mathbf{C}(h)$ corresponding to x .

- (3) Given any family \hat{h} in $\mathcal{M}^{\infty,1}$ containing h and one of the refinement maps $\Phi : \mathbf{C}(h) \longrightarrow \mathbf{C}(f)$ from property 2 above, the third property that h must satisfy to be in \mathcal{O} is that there exists an extension $\hat{\Phi}$ of Φ to some neighborhood of $\mathbf{C}(h) \subset \mathbf{C}(\hat{h})$

$$\begin{array}{ccc} \mathbf{C}(\hat{h}) & \xrightarrow{\hat{\Phi}} & \mathbf{C}(\hat{f}) \\ \downarrow & & \downarrow \\ \mathbf{F}(\hat{h}) & \longrightarrow & \mathbf{F}(\hat{f}) \end{array}$$

so that \hat{h} is equal to $\hat{f} \circ \hat{\Phi}$ followed by exponentiation of some section of $(\hat{f} \circ \hat{\Phi})^* T_{vert} \hat{\mathbf{B}}$ which vanishes at the extra n marked points on $\mathbf{C}(h)$ corresponding to x .

Claim 5.3. *If f is in \hat{f} , then every refinement of f is in \mathcal{O} .*

Criteria 6 and 4 imply that property 1 holds for every curve f in \hat{f} . As refinement does not affect transversal intersections, it follows that property 1 also holds for any refinement of f . Property 2 also holds trivially for any refinement of f using the zero vector field. To see that property 3 holds, note that \hat{f} has universal tropical structure, and apply Lemma 4.3. This implies that given any family of curves \hat{h} containing a refinement h of f , restricted to some neighborhood of h in \hat{h} , there exists a map

$$\begin{array}{ccc} \mathbf{C}(\hat{h}) & \xrightarrow{\hat{\Phi}} & \mathbf{C}(\hat{f}) \\ \downarrow & & \downarrow \\ \mathbf{F}(\hat{h}) & \longrightarrow & \mathbf{F}(\hat{f}) \end{array}$$

which is the refinement map restricted to $\mathbf{C}(h) \subset \mathbf{C}(\hat{h})$, and so that the distance between $\hat{f} \circ \hat{\Phi}$ and \hat{h} is bounded. In particular, this implies that we may choose $\hat{\Phi}$ so that \hat{h} is $\hat{f} \circ \hat{\Phi}$ followed by exponentiation of some section of $(\hat{f} \circ \hat{\Phi})^* T_{\text{vert}} \hat{\mathbf{B}}$ which vanishes on $\mathbf{C}(h) \subset \mathbf{C}(\hat{h})$. Therefore property 3 holds.

Claim 5.4. *If properties 1 and 2 hold for a curve h , then property 3 holds for h .*

Let \hat{h} be a family of curves containing h . As properties 1 and 2 hold, for a given intersection x from property 1, there exists a map

$$\Phi : \mathbf{C}(h) \longrightarrow \mathbf{C}(\hat{f})$$

which is a refinement map onto the domain of some curve in \hat{f} so that h is equal to $\hat{f} \circ \Phi$ followed by exponentiating some vectorfield which is a section of $(\hat{f} \circ \Phi)^* T_{\text{vert}} \hat{\mathbf{B}}$ which vanishes at the extra n marked points on $\mathbf{C}(h)$ corresponding to x . Equivalently, $\hat{f} \circ \Phi$ is equal to h composed with exponentiation of some section ν of $h^* T_{\text{vert}} \hat{\mathbf{B}}$ which vanishes on the extra marked points. Define h' to be $\hat{f} \circ \Phi$, and define \hat{h}' to be \hat{h} composed with exponentiation of some extension of ν . As h' is a refinement of a curve in \hat{f} , Claim 5.3 implies that property 3 holds for h' . Therefore, on a neighborhood of $\mathbf{C}(h)$, there is a map

$$\hat{\Phi} : \mathbf{C}(\hat{h}') = \mathbf{C}(\hat{h}) \longrightarrow \mathbf{C}(\hat{f})$$

extending $\Phi : \mathbf{C}(h) = \mathbf{C}(h') \longrightarrow \mathbf{C}(\hat{f})$ so that \hat{h}' is equal to $\hat{f} \circ \hat{\Phi}$ followed by exponentiation of some section of $(\hat{f} \circ \hat{\Phi})^* T_{\text{vert}} \hat{\mathbf{B}}$ which vanishes on the extra marked points on $\mathbf{C}(\hat{h})$. As \hat{h}' is obtained from \hat{h} by exponentiating a section of $\hat{h}^* T_{\text{vert}} \hat{\mathbf{B}}$ vanishing on the extra marked points on $\mathbf{C}(h')$, the same property holds for h and \hat{h} . In other words, property 3 holds for h . This completes the proof of Claim 5.4.

Claim 5.5. *If h is in \mathcal{O} , and \hat{h} is any family containing h , property 1 holds on a neighborhood of h in \hat{h} .*

Choose a point x in $\mathbf{F}(h^{+n}) \subset \mathbf{F}(\hat{h}^{+n})$ in the inverse image of the image of $ev^{+n}(\hat{f}) \circ s$. We must show that this transverse intersection point x continues to exist in a neighborhood of h . Property 3 and criterion 4 imply that on an open neighborhood of x in $\mathbf{F}(\hat{h}^{+n})$, $ev^{+n}(\hat{h})$ factors as a map to $\mathbf{F}(\hat{f}^{+n})$ followed by $ev^{+n}(\hat{f})$ so that x is sent to the image of $s : \mathbf{F}(\hat{f}) \longrightarrow \mathbf{F}(\hat{f}^{+n})$. It follows that each of these transverse intersection points continues to exist for curves in a neighborhood of h within \hat{h} . In a neighborhood of h within \hat{h} , there will be no extra intersection points with the closure of the image of $ev^{+n}(\hat{f}) \circ s$. It follows that property 1 holds for curves in a neighborhood of h within \hat{h} , and Claim 5.5 is proved.

Claim 5.6. *If \hat{h} contains a curve h in \mathcal{O} , then every curve in an open neighborhood of h within \hat{h} satisfies property 2.*

We shall complete the proof of Claim 5.6 only near the end of the proof of this theorem. For now, note that Claim 5.4 together with Claims 5.5 and 5.6 imply that given any family \hat{h} of curves, the curves which are in \mathcal{O} form an open subset. It follows from Corollary 2.8 that \mathcal{O} is open.

Given a family of curves \hat{h}_0 , we shall now prove some facts for a sufficiently small open neighborhood \hat{h} of the curves in \hat{h}_0 which are also in \mathcal{O} . This will serve to prove Claim 5.6 at the same time as we prove facts about any family of curves \hat{h} within \mathcal{O} .

By Claim 5.5, we may assume that property 1 holds for all curves in \hat{h} , so the following fiber product is transverse, and comes with an equidimensional submersion of degree $|G|$ to $\mathbf{F}(\hat{h})$.

$$(4) \quad \begin{array}{ccc} \mathbf{F}(\hat{h}^{+n}) & \longleftarrow \mathbf{F}(\hat{h}^{+n}) \times_{ev^{+n}(\hat{h}) \times_{ev^{+n}(\hat{f}) \circ s} \mathbf{F}(\hat{f})} & \longrightarrow \mathbf{F}(\hat{f}) \\ & \searrow & \downarrow \\ & & \mathbf{F}(\hat{h}) \end{array}$$

As noted above, if h is in \mathcal{O} , property 3 and criterion 4 imply that for each intersection point x of $ev^{+n}(h)$ with $ev^{+n}(\hat{f}) \circ s$, on an open neighborhood of x in $\mathbf{F}(\hat{h}^{+n})$, $ev^{+n}(\hat{h})$ factors as a map to $\mathbf{F}(\hat{f}^{+n})$ followed by $ev^{+n}(\hat{f})$ sending x to the image of s . It follows we may assume that the above map is a $|G|$ -fold covering map (so it is locally an isomorphism onto an open subset of $\mathbf{F}(\hat{h})$, instead of just being locally a refinement of an open subset of $\mathbf{F}(\hat{h})$).

Criterion 3 implies that there is an action of G on $\mathcal{M}(pt) \times \hat{\mathbf{B}}_{\mathbf{B}_0}^n$ which permutes the labels of the extra n marked points and the corresponding factors of $\hat{\mathbf{B}}_{\mathbf{B}_0}^n$ so that $ev^{+n}(\hat{f}) \circ s$ is G -equivariant. As the image of $ev^{+n}(\hat{h})$ automatically contains all the results of such a permutation of marked points, this G action gives an action of G on the above fiber product in (4). This makes the above $|G|$ -fold cover of $\mathbf{F}(\hat{h})$ into a G -bundle because the action on the image of $ev^{+n}(\hat{f}) \circ s$ simply permutes the marked points, so each G -orbit is contained within the same fiber of $\mathbf{F}(\hat{h}^{+n}) \rightarrow \mathbf{F}(\hat{h})$. Therefore, the above map from our G -bundle to $\mathbf{F}(\hat{f})$ is equivalent to a map from $\mathbf{F}(\hat{h})$ to $\mathbf{F}(\hat{f})/G$.

Define a G -fold cover \hat{h}' of the family \hat{h} by setting

$$\mathbf{F}(\hat{h}') := \mathbf{F}(\hat{h}^{+n}) \times_{ev^{+n}(\hat{h}) \times_{ev^{+n}(\hat{f}) \circ s} \mathbf{F}(\hat{f})}$$

and pulling back the family \hat{h} by

$$\begin{array}{ccccc} & & \hat{h}' & & \\ & \nearrow & & \searrow & \\ \mathbf{C}(\hat{h}') & \longrightarrow & \mathbf{C}(\hat{h}) & \xrightarrow{\hat{h}} & \hat{\mathbf{B}} \\ \downarrow & & \downarrow & & \\ \mathbf{F}(\hat{h}') & \longrightarrow & \mathbf{F}(\hat{h}) & & \end{array}$$

Given any family of curves \hat{g} in \mathcal{O} and a map $\hat{g} \rightarrow \hat{h}$, Lemma 4.11 implies that there is a naturally induced map $\hat{g}^{+n} \rightarrow \hat{h}^{+n}$, which induces a G -equivariant map

$$\mathbf{F}(\hat{g}^{+n}) \times_{ev^{+n}(\hat{g}) \times_{ev^{+n}(\hat{f}) \circ s} \mathbf{F}(\hat{f})} \longrightarrow \mathbf{F}(\hat{h}^{+n}) \times_{ev^{+n}(\hat{h}) \times_{ev^{+n}(\hat{f}) \circ s} \mathbf{F}(\hat{f})}$$

Therefore, we have a canonical G -equivariant lift of the map $\hat{g} \longrightarrow \hat{h}$

$$\begin{array}{ccc} \hat{g}' & \longrightarrow & \hat{h}' \\ \downarrow & & \downarrow \\ \hat{g} & \longrightarrow & \hat{h} \end{array}$$

so that the G -equivariant map $\mathbf{F}(\hat{g}') \longrightarrow \mathbf{F}(\hat{f})$ factorizes as $\mathbf{F}(\hat{g}') \longrightarrow \mathbf{F}(\hat{f}') \longrightarrow \mathbf{F}(\hat{f})$. In other words, we have constructed a map $\mathcal{O} \longrightarrow \mathbf{F}(\hat{f})/G$. To check Criterion 1 of the definition of a core family on page 37, we should check what this map is applied to \hat{f} itself.

$$\mathbf{F}(\hat{f}') := \mathbf{F}(\hat{f}^{+n})_{ev^{+n}(\hat{f}) \times_{ev^{+n}(\hat{f}) \circ s} \mathbf{F}(\hat{f})}$$

$\mathbf{F}(\hat{f}')$ may be regarded as the family of pairs (f_1, f_2) where f_2 is a curve in \hat{f} , and f_1 is a curve in \hat{f}^{+n} isomorphic to $s^* f_2^{+n}$. As $s^* f_2^{+n}$ has no automorphisms, there is a canonical isomorphism of $s^* f_2^{+n}$ with f_1 , so the choice of f_1 is equivalent to a choice of curve f_1' in \hat{f} and an element g of G so that $g * f_2 = f_1'$. The map

$$\mathbf{F}(\hat{f}') \longrightarrow \mathbf{F}(\hat{f})$$

is the map which sends (f_1', g, f_2) to f_2 , and the G -fold cover map $\hat{f}' \longrightarrow \hat{f}$ corresponds to the map $(f_1', g, f_2) \mapsto f_1'$. These maps fit into the following commutative diagram

$$\begin{array}{ccc} \mathbf{F}(\hat{f}') & \longrightarrow & \mathbf{F}(\hat{f}) \\ \downarrow & & \downarrow \\ \mathbf{F}(\hat{f}) & \longrightarrow & \mathbf{F}(\hat{f})/G \end{array}$$

where the top arrow is the map from $\mathcal{O} \longrightarrow \mathbf{F}(\hat{f})/G$, the left arrow is the G -fold cover map, and the other two arrows are both the quotient map $\mathbf{F}(\hat{f}) \longrightarrow \mathbf{F}(\hat{f})/G$. It follows that the map $\mathcal{O} \longrightarrow \mathbf{F}(\hat{f})/G$ applied to \hat{f} is the quotient map $\mathbf{F}(\hat{f}) \longrightarrow \mathbf{F}(\hat{f})/G$, as required by Criterion 1 of Definition 5.1.

We must lift our map $\mathcal{O} \longrightarrow \mathbf{F}(\hat{f})/G$ to a fiberwise holomorphic map

$$\begin{array}{ccc} \mathcal{O}^{+1} & \longrightarrow & \mathbf{C}(\hat{f})/G \\ \downarrow & & \downarrow \\ \mathcal{O} & \longrightarrow & \mathbf{F}(\hat{f})/G \end{array}$$

In particular, we must construct a lift of $\mathbf{F}(\hat{h}') \longrightarrow \mathbf{F}(\hat{f})$ to a G -equivariant, fiberwise holomorphic map

$$\begin{array}{ccc} \mathbf{C}(\hat{h}') & \xrightarrow{\Phi} & \mathbf{C}(\hat{f}) \\ \downarrow & & \downarrow \\ \mathbf{F}(\hat{h}') & \longrightarrow & \mathbf{F}(\hat{f}) \end{array}$$

so that \hat{h}' is equal to $\hat{f} \circ \Phi$ when restricted to the pullback under Φ of each of the sections s_i .

Consider the map

$$(5) \quad \tilde{ev}^{+n}(\hat{f}) : \mathbf{C}(\hat{f}^{+n}) \longrightarrow \mathcal{M}(pt) \times \hat{\mathbf{B}}_{\mathbf{B}_0}^n$$

constructed from the composition of $ev^{+(n+1)}(\hat{f})$ with a projection map

$$\begin{array}{ccc} & & \mathcal{M}(pt) \times \hat{\mathbf{B}}_{\mathbf{B}_0}^{n+1} \\ & \nearrow^{ev^{+(n+1)}(\hat{f})} & \downarrow \\ \mathbf{C}(\hat{f}^{+n}) & & \mathcal{M}(pt) \times \hat{\mathbf{B}}_{\mathbf{B}_0}^n \\ & \searrow_{\tilde{ev}^{+n}(\hat{f})} & \end{array}$$

forgetting the image of the $(n + 1)$ st marked point. On the second component, $\tilde{ev}^{+n}(\hat{f})$ is equal to the composition of the projection $\mathbf{C}(\hat{f}^{+n}) \rightarrow \mathbf{F}(\hat{f}^{+n})$ with the map $\hat{f}^{+(n-1)}$.

Denote by $s^*\hat{f}^{+n}$ the pullback of \hat{f}^{+n} under the map $s : \mathbf{F}(\hat{f}) \rightarrow \mathbf{F}(\hat{f}^{+n})$. Criteria 2 and 4 imply that the above evaluation map $\tilde{ev}^{+n}(\hat{f})$ is an equidimensional embedding in a neighborhood of $\mathbf{C}(s^*\hat{f}^{+n}) \subset \mathbf{C}(\hat{f}^{+n})$, and that the following is a pullback diagram of families of curves

$$\begin{array}{ccc} \mathbf{C}(\hat{f}^{+n}) & \xrightarrow{\tilde{ev}^{+n}(\hat{f})} & \mathcal{M}(pt)^{+1} \times \hat{\mathbf{B}}_{\mathbf{B}_0}^n \\ \downarrow & & \downarrow \\ \mathbf{F}(\hat{f}^{+n}) & \xrightarrow{ev^{+n}(\hat{f})} & \mathcal{M}(pt) \times \hat{\mathbf{B}}_{\mathbf{B}_0}^n \end{array}$$

We can therefore regard $\mathbf{C}(s^*\hat{f}^{+n})$ as a fiber product

$$\begin{array}{ccc} \mathbf{C}(s^*\hat{f}^{+n}) & \longrightarrow & \mathcal{M}(pt)^{+1} \times \hat{\mathbf{B}}_{\mathbf{B}_0}^n \\ \downarrow & & \downarrow \\ \mathbf{F}(\hat{f}) & \xrightarrow{ev^{+n}(\hat{f}) \circ s} & \mathcal{M}(pt) \times \hat{\mathbf{B}}_{\mathbf{B}_0}^n \end{array}$$

In order to define $\Phi : \mathbf{C}(\hat{h}') \rightarrow \mathbf{C}(\hat{f})$, we shall first define a map $\tilde{\Phi} : \mathbf{C}' \rightarrow \mathbf{C}(s^*\hat{f}^{+n})$, where \mathbf{C}' shall be obtained from $\mathbf{C}(\hat{h}')$ by adding n extra external edges.

Define $\mathbf{C}' \rightarrow \mathbf{F}(\hat{h}')$ as the pullback of $\mathbf{C}(\hat{h}^{+n}) \rightarrow \mathbf{F}(\hat{h}^{+n})$ under the inclusion $\mathbf{F}(\hat{h}') \hookrightarrow \mathbf{F}(\hat{h}^{+n})$. (This \mathbf{C}' is $\mathbf{C}(\hat{h}')$ with n extra external edges.)

$$\begin{array}{ccc} \mathbf{C}' & \hookrightarrow & \mathbf{C}(\hat{h}^{+n}) \\ \downarrow & & \downarrow \\ \mathbf{F}(\hat{h}') & \hookrightarrow & \mathbf{F}(\hat{h}^{+n}) \end{array}$$

The action of G on $\mathbf{F}(\hat{h}') \subset \mathbf{F}(\hat{h}^{+n})$ is some permutation of marked points. This G action extends to a G action on $\mathbf{F}(\hat{h}^{+n})$ permuting these marked points, and lifts to a G action on $\mathbf{C}(\hat{h}^{+n})$ which just permutes the same marked points (which are external edges of the curves in $\mathbf{C}(\hat{h}^{+n})$). Therefore, we have a lift of our G action on $\mathbf{F}(\hat{h}')$ to a G action on \mathbf{C}' .

Claim 5.7. *There exists a unique fiberwise holomorphic map*

$$\begin{array}{ccc} \mathbf{C}' & \xrightarrow{\tilde{\Phi}} & \mathbf{C}(s^*\hat{f}^{+n}) \\ \downarrow & & \downarrow \\ \mathbf{F}(\hat{h}') & \longrightarrow & \mathbf{F}(\hat{f}) \end{array}$$

so that $\tilde{ev}^{+n}(\hat{f}) \circ \tilde{\Phi} = \tilde{ev}^{+n}(\hat{h})$ on \mathbf{C}' .

(The uniqueness condition in Claim 5.7 above makes no mention of lifting our given map $\mathbf{F}(\hat{h}') \rightarrow \mathbf{F}(\hat{f})$, however, any map $\tilde{\Phi}$ satisfying this condition is automatically the lift of our already constructed map $\mathbf{F}(\hat{h}') \rightarrow \mathbf{F}(\hat{f})$.)

To prove Claim 5.7, consider the commutative diagram

$$\begin{array}{ccc} \mathbf{C}' & \xrightarrow{\tilde{ev}^{+n}(\hat{h})} & \mathcal{M}(pt)^{+1} \times \hat{\mathbf{B}}_{\mathbf{B}_0}^n \\ \downarrow & & \downarrow \\ \mathbf{F}(\hat{h}') & \xrightarrow{ev^{+n}(\hat{h})} & \mathcal{M}(pt) \times \hat{\mathbf{B}}_{\mathbf{B}_0}^n \end{array}$$

where the bottom arrow can either be regarded as

$$\mathbf{F}(\hat{h}') \hookrightarrow \mathbf{F}(\hat{h}^{+n}) \xrightarrow{ev^{+n}(\hat{h})} \mathcal{M}(pt) \times \hat{\mathbf{B}}_{\mathbf{B}_0}^n$$

or as

$$\mathbf{F}(\hat{h}') \longrightarrow \mathbf{F}(\hat{f}) \xrightarrow{ev^{+n}(\hat{f}) \circ s} \mathcal{M}(pt) \times \hat{\mathbf{B}}_{\mathbf{B}_0}^n$$

Therefore, by regarding $\mathbf{C}(s^* \hat{f}^{+n})$ as a fiber product, we get a commutative diagram

$$\begin{array}{ccccc} & & \tilde{ev}^{+n}(\hat{h}) & & \\ & \nearrow & & \searrow & \\ \mathbf{C}' & \xrightarrow{\tilde{\Phi}} & \mathbf{C}(s^* \hat{f}^{+n}) & \xrightarrow{\tilde{ev}^{+n}(\hat{f})} & \mathcal{M}(pt)^{+1} \times \hat{\mathbf{B}}_{\mathbf{B}_0}^n \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{F}(\hat{h}') & \longrightarrow & \mathbf{F}(\hat{f}) & \xrightarrow{ev^{+n}(\hat{f}) \circ s} & \mathcal{M}(pt) \times \hat{\mathbf{B}}_{\mathbf{B}_0}^n \\ & \searrow & & \nearrow & \\ & & ev^{+n}(\hat{h}) & & \end{array}$$

The map $\tilde{\Phi}$ above is a fiberwise holomorphic map from \mathbf{C}' to $\mathbf{C}(s^* \hat{f}^{+n})$ so that $\tilde{ev}^{+n}(\hat{f}) \circ \tilde{\Phi} = \tilde{ev}^{+n}(\hat{h})$ on \mathbf{C}' . Any map $\tilde{\Phi}$ satisfying this condition must fit into the commutative diagram above. The bottom loop of the above diagram is uniquely satisfied by our given map $\mathbf{F}(\hat{h}') \longrightarrow \mathbf{F}(\hat{f})$, therefore such a map $\tilde{\Phi}$ is unique. This completes the proof of Claim 5.7.

Consider $\tilde{\Phi}$ in the case that \hat{h} is a curve h in \mathcal{O} . Then property 2 of \mathcal{O} implies that there exists a map $\Phi : \mathbf{C}(h') \longrightarrow \mathbf{C}(\hat{f})$, which restricted to each curve in h' is a holomorphic refinement map onto a fiber of $\mathbf{C}(\hat{f})$ so that $\hat{f} \circ \Phi$ is equal to h' restricted to Φ^{-1} of the extra marked points on $\mathbf{C}(\hat{f})$. It follows that Φ lifts to a holomorphic refinement map $\mathbf{C}' \longrightarrow \mathbf{C}(s^* \hat{f}^{+n})$ satisfying the conditions required of $\tilde{\Phi}$. Therefore, the uniqueness of $\tilde{\Phi}$ implies that $\tilde{\Phi}$ restricted to any fiber in \mathcal{O} is a holomorphic refinement map.

To construct Φ from $\tilde{\Phi}$, we may assume that we have restricted \hat{h} to curves in an open neighborhood of \mathcal{O} so that $\tilde{\Phi}$ is fiberwise a holomorphic refinement map. The extra external edges on \mathbf{C}' are just the pullback of the extra external edges on $\mathbf{C}(s^* \hat{f}^{+n})$, which are all distinct and contained in smooth components of $\mathbf{C}(\hat{f})$. It follows that we may forget these extra external edges in the domain and target of $\tilde{\Phi}$ to obtain a $C^{\infty,1}$ map Φ .

$$\begin{array}{ccc} \mathbf{C}' & \xrightarrow{\tilde{\Phi}} & \mathbf{C}(s^* \hat{f}^{+n}) \\ \downarrow & & \downarrow \\ \mathbf{C}(\hat{h}') & \xrightarrow{\Phi} & \mathbf{C}(\hat{f}) \\ \downarrow & & \downarrow \\ \mathbf{F}(\hat{h}') & \longrightarrow & \mathbf{F}(\hat{f}) \end{array}$$

which is a holomorphic refinement map restricted to each fiber.

As noted above, the uniqueness of $\tilde{\Phi}$ implies that restricted to any curve h in \mathcal{O} , Φ must be the map from property 2 of \mathcal{O} . It follows that on a neighborhood in \hat{h}' of all these curves in \mathcal{O} , there exists some $C^{\infty, \perp}$ section v of $(\Phi \circ \hat{f})^* T_{\text{vert}} \hat{\mathbf{B}}$ which vanishes on all the extra marked points so that \hat{h}' is equal to $\hat{f} \circ \Phi$ followed by exponentiation of v . In particular, property 2 holds on this neighborhood and Claim 5.6 has now been proved, so \mathcal{O} is open. We have also just proved that Criterion 2 of definition 5.1 holds.

The requirement that $\tilde{e}v^{+n}(\hat{f}) \circ \tilde{\Phi} = \tilde{e}v^{+n}(\hat{h})$ on \mathbf{C}' is equivalent to the requirement that restricted to the inverse image of the extra marked points, $\hat{f} \circ \Phi$ is equal to \hat{h} . As any choice of Φ satisfying these conditions is equivalent to a choice of $\tilde{\Phi}$, the uniqueness of $\tilde{\Phi}$ implies Φ is the unique map $\mathbf{C}(\hat{h}') \rightarrow \mathbf{C}(\hat{f})$ which is a holomorphic refinement map restricted to each fiber so that $\hat{f} \circ \Phi$ is equal to \hat{h}' restricted to the inverse image of the extra marked points $\{s_i\}$.

The uniqueness of Φ and the G -equivariant nature of the conditions imposed on it imply that Φ is a G -equivariant map, so Φ may be regarded as a map from $\mathbf{C}(\hat{h})$ to $\mathbf{C}(\hat{f})/G$. The fact that $\mathcal{O} \rightarrow \mathbf{F}(\hat{f})/G$ is a map of stacks and the uniqueness of this lift Φ imply that Φ defines a fiberwise holomorphic map

$$\begin{array}{ccc} \mathcal{O}^{+1} & \xrightarrow{\Phi} & \mathbf{C}(\hat{f})/G \\ \downarrow & & \downarrow \\ \mathcal{O} & \longrightarrow & \mathbf{F}(\hat{f})/G \end{array}$$

In particular, given any map of families $\hat{g} \rightarrow \hat{h}$, the map of G -fold covers $\hat{g}' \rightarrow \hat{h}'$ is compatible with Φ in the sense that $\Phi : \mathbf{C}(\hat{g}') \rightarrow \mathbf{C}(\hat{f})$ factorizes as

$$\begin{array}{ccccc} & & \Phi & & \\ & \searrow & \text{---} & \nearrow & \\ \mathbf{C}(\hat{g}') & \longrightarrow & \mathbf{C}(\hat{h}') & \xrightarrow{\Phi} & \mathbf{C}(\hat{f}) \end{array}$$

As we have already checked Criterion 1 for the map $\mathcal{O} \rightarrow \mathbf{F}(\hat{f})/G$, the uniqueness of this lift Φ also implies Criterion 1 from Definition 5.1, namely, Φ applied to \hat{f} corresponds to the quotient map $\mathbf{C}(\hat{f}) \rightarrow \mathbf{C}(\hat{f})/G$. The uniqueness property of Φ implies that Φ is unaffected by flowing by vectorfields which vanish at the extra marked points. Therefore Criterion 1 implies Criterion 3 of Definition 5.1. \square

The following proposition constructs a core family containing a given stable holomorphic curve which has at least one smooth component (so its domain is not \mathbf{T}).

Proposition 5.8. *Given a curve f in \mathcal{M}^{st} with a domain which is not equal to \mathbf{T} , and a collection of marked points $\{p_j\}$ in the interior of the smooth components of $\mathbf{C}(f)$, there exists a $C^{\infty, \perp}$ core family $(\hat{f}/G, \{s_i\})$ satisfying the requirements of Theorem 5.2, with \hat{f} a family containing f so that the restriction of $\{s_i\}$ to f contains the given marked points $\{p_j\}$.*

Proof: Theorem 4.2 allows us to construct a family of curves \hat{f}' with universal tropical structure containing f with a finite group G' of automorphisms which act freely and transitively on the set of maps of f into \hat{f}' . The smooth part of the strata of $\mathbf{F}(\hat{f}')$ containing f consists of a single point, and G' is a subgroup of the group G of automorphisms of $[f]$. We may also assume that there is only one curve in $[\hat{f}']$ which is isomorphic to $[f]$.

Let \hat{f}_0 be the quotient of $G \times \hat{f}'$ by the equivalence relation setting (g, \hat{f}') equal to $(gh^{-1}, h * \hat{f}')$ for any $h \in G'$. In other words, \hat{f}_0 is $|G/G'|$ disjoint copies of \hat{f}' . G acts as a group of automorphisms on \hat{f}_0 by multiplying the G factor of $G \times \hat{f}'$ on the left. This G action is free and transitive on the set of maps of f into \hat{f}_0 . This new family \hat{f}_0 also has universal tropical structure because having universal tropical structure is a locally defined condition.

Choose one particular inclusion of f into \hat{f}_0 , then choose a G -invariant collection of n non intersecting sections s_i of $\mathbf{C}(\hat{f}_0) \rightarrow \mathbf{F}(\hat{f})$ so that the intersection of these sections s_i with $\mathbf{C}(f) \subset \mathbf{C}(\hat{f}_0)$ correspond to a set of marked points $\{p_i\}$ so that the following conditions hold:

- (1) These marked points $\{p_i\}$ contain the set of marked points given in the statement of the theorem.
- (2) Each p_i is in a smooth component of $\mathbf{C}(f)$.
- (3) The action of G as the automorphism group of $[f]$ permutes the marked points p_i , and the action of G on the set of sections s_i is compatible in the sense that if g as an automorphism of $[\mathbf{C}(f)]$ sends p_i to p_j , then the action of g on $\mathbf{F}(\hat{f}_0)$ followed by s_j is equal to s_i followed by the action of g on $\mathbf{C}(\hat{f}_0)$. Representing the various actions of g simply as g , we may write this condition as

$$g * p_i = p_j \quad \text{implies that} \quad s_j \circ g = g \circ s_i$$

- (4) $\mathbf{C}(f)$ with the set of points p_i so that $d[f]$ is injective at p_i is stable.
- (5) The nodal Riemann surface $[\mathbf{C}(f)]$ with the extra marked points $\{p_i\}$ has no automorphisms.
- (6) There is at least one marked point on each smooth component of $\mathbf{C}(f)$.

Clearly, items 2, 4, 5 and 6 above remain true for the marked points obtained by intersecting $\{s_i\}$ with $\mathbf{C}(f')$ for f' in a neighborhood of f .

Following the notation of the proof of Theorem 5.2, let $s : \mathbf{F}(\hat{f}_0) \rightarrow \mathbf{F}(\hat{f}_0^{+n})$ be the map determined by the n sections, $\{s_i\}$, so the domain of the family of curves $s^* \hat{f}_0^{+n}$ is $\mathbf{C}(\hat{f}_0)$ with extra external edges at the images of s_i .

Claim 5.9. *If a curve h in f^{+n} has smooth part isomorphic to the smooth part of a curve in $s^* \hat{f}_0^{+n}$, then h is actually isomorphic to a curve in $s^* \hat{f}_0^{+n}$.*

To prove Claim 5.9, consider the corresponding isomorphism of $[f]$ with a curve in $[\hat{f}_0]$. As there was only one curve in $[\hat{f}']$ isomorphic to $[f]$, this isomorphism must decompose as an automorphism g_1 of $[f]$ followed by our chosen inclusion $[f] \rightarrow [\hat{f}_0]$, followed by the action of some $g_2 \in G$. Item 3 implies that the pullback of $[s_i]$ under such an isomorphism is equal to the pullback of $[s_i]$ under our chosen inclusion $[f] \rightarrow [\hat{f}_0]$ followed by the action of $g_2 g_1$. Therefore, the extra marked points on h may be obtained by pulling back the sections s_i via our chosen inclusion of f into \hat{f}_0 followed by the action of an element of G . Therefore, h is isomorphic to a curve in $s^* \hat{f}_0^{+n}$, and Claim 5.9 is true.

Item 5 implies that each of the $|G|$ inclusions of f into \hat{f}_0 corresponds to a different intersection of $ev^{+n}(f)$ with the image of $ev^{+n}(\hat{f}_0) \circ s$. Claim 5.9 implies that no other curve in f^{+n} has the same smooth part as a curve in $s^* \hat{f}_0^{+n}$. We may therefore add extra marked points satisfying the above properties until $[ev^{+n}(f)]$ has precisely $|G|$ intersections with the image of $[ev^{+n}(\hat{f}_0) \circ s]$.

Consider the tropical structure of the map

$$ev^{+n}(\hat{f}_0) \circ s : \mathbf{F}(\hat{f}_0) \longrightarrow \mathcal{M}(pt) \times \hat{\mathbf{B}}_{\mathbf{B}_0}^n$$

at the curve f . This is some integral affine map $P_u \longrightarrow P'$. This tropical structure records the image of $\hat{f}_0 \circ s_i$ in $\hat{\mathbf{B}}$, and because of item 5 above, it also records the length of the internal edges of curves in \hat{f}_0 . Because \hat{f}_0 has universal tropical structure, Remark 3.3 of [15] implies that the map $P_u \longrightarrow P'$ is injective and sends integral vectors on P_u to a full sublattice of the integral vectors on P' . In other words, $ev^{+n}(\hat{f}_0) \circ s$ sends integral vectors in its domain to a full sublattice of the integral vectors in its target.

As the smooth part of the strata of $\mathbf{F}(\hat{f}_0)$ containing f is 0-dimensional, $T_f \mathbf{F}(\hat{f}_0)$ consists of only integral vectors. The above discussion then implies that if \hat{f}_0 is chosen small enough, $ev^{+n}(\hat{f}_0) \circ s$ is injective and has injective derivative.

Item 4 above ensures that at the point $s(f) \in \mathbf{F}(\hat{f}_0^{+n})$, the derivative of the smooth part of

$$ev^{+n}(f) : \mathbf{F}(f^{+n}) \longrightarrow \mathcal{M}(pt) \times \hat{\mathbf{B}}_{\mathbf{B}_0}^n$$

is injective. As each of the p_i are distinct and on smooth components of $\mathbf{C}(f)$, a neighborhood of $s(f)$ in $\mathbf{F}(f^{+n})$ is isomorphic to \mathbb{R}^{2n} . Therefore, the derivative of $ev^{+n}(f)$ at this point is injective and has no nontrivial integral vectors in its image. It follows that if \hat{f}_0 is chosen small enough, restricted to a neighborhood of the image of s ,

$$ev^{+n}(\hat{f}_0) : \mathbf{F}(\hat{f}_0^{+n}) \longrightarrow \mathcal{M}(pt) \times \hat{\mathbf{B}}_{\mathbf{B}_0}^n$$

is injective and has injective derivative.

We shall now extend \hat{f}_0 and $\{s_i\}$ to a G -invariant family \hat{f} with a G -invariant set of sections $\{s_i\}$ so that $ev^{+n}(\hat{f})$ is an equidimensional embedding in a neighborhood of the image of s , (so Criterion 4 of Theorem 5.2 will be satisfied.)

The action of G permutes the n sections $\{s_i\}$. In other words, there is an action of G on \hat{f}_0^{+n} which lifts the action of G on \hat{f}_0 and permutes the labels of the extra edges in \hat{f}_0^{+n} so that $s : \mathbf{F}(\hat{f}_0) \longrightarrow \mathbf{F}(\hat{f}_0^{+n})$ is G -equivariant.

There is a corresponding action of G on $\mathcal{M}(pt) \times \hat{\mathbf{B}}_{\mathbf{B}_0}^n$ permuting the labels of the n extra external edges of curves in $\mathcal{M}(pt)$ and the n factors of $\hat{\mathbf{B}}_{\mathbf{B}_0}^n$. With this action of G ,

$$ev^{+n}(\hat{f}_0) : \mathbf{F}(\hat{f}_0^{+n}) \longrightarrow \mathcal{M}(pt) \times \hat{\mathbf{B}}_{\mathbf{B}_0}^n$$

is also G -equivariant. (This map $ev^{+n}(\hat{f}_0)$ is invariant under the lift of the action of G on \hat{f}_0 , and obviously equivariant under the action of G which permutes the same n labels in the domain and target.)

Choose a G -invariant metric on $\mathcal{M}(pt) \times \hat{\mathbf{B}}_{\mathbf{B}_0}^n$, and let U be some small G -invariant tubular neighborhood of the image of $ev^{+n}(\hat{f}_0)$ restricted to a neighborhood of the section s . (Note that in general U will not be an open neighborhood in the usual topology on $\mathcal{M}(pt) \times \hat{\mathbf{B}}_{\mathbf{B}_0}^n$, as the topology induced by a metric on an exploded manifold is finer than the usual topology.) Let V be the restriction of this disk bundle U to the image of $ev^{+n}(\hat{f}_0) \circ s$, (so V has codimension $2n$ in U .) As $ev^{+n}(\hat{f}_0)$ and $ev^{+n}(\hat{f}_0) \circ s$ are G -equivariant and the metric used to define our tubular neighborhood is G -invariant, V is G -invariant.

Define $\mathbf{F}(\hat{f})$ to be V , and define $\mathbf{C}(s^* \hat{f})$ by the pullback diagram

$$\begin{array}{ccc}
\mathbf{C}(s^*\hat{f}) & \longrightarrow & \mathcal{M}(pt)^{+1} \times \hat{\mathbf{B}}_{\mathbf{B}_0}^n \\
\downarrow & & \downarrow \\
\mathbf{F}(\hat{f}) & \hookrightarrow & \mathcal{M}(pt) \times \hat{\mathbf{B}}_{\mathbf{B}_0}^n
\end{array}$$

The action of G on $\mathcal{M}(pt) \times \hat{\mathbf{B}}_{\mathbf{B}_0}^n$ lifts to an action of G on $\mathcal{M}(pt)^{+1} \times \hat{\mathbf{B}}_{\mathbf{B}_0}^n$ which permutes the same external edge labels. Therefore, there is an action of G on the family of curves $\mathbf{C}(s^*\hat{f}) \rightarrow \mathbf{F}(\hat{f})$ which makes the above diagram G -equivariant. By removing the extra edges in $\mathbf{C}(s^*\hat{f})$, and remembering their location with sections s_i , we get a G -invariant family of curves $\mathbf{C}(\hat{f}) \rightarrow \mathbf{F}(\hat{f})$ with a G -invariant set of sections $\{s_i\}$.

Note that there is a G -equivariant inclusion of $\mathbf{C}(\hat{f}_0)$ as a subfamily of $\mathbf{C}(\hat{f})$ via the diagram

$$\begin{array}{ccc}
\mathbf{C}(\hat{f}_0) & \longrightarrow & \mathbf{C}(\hat{f}) \\
\downarrow & & \downarrow \\
\mathbf{F}(\hat{f}_0) & \xrightarrow{ev^{+n}(\hat{f}_0) \circ s} & V = \mathbf{F}(\hat{f})
\end{array}$$

Note also that the restriction of the sections s_i of $\mathbf{C}(\hat{f}) \rightarrow \mathbf{F}(\hat{f})$ are our original sections s_i . As $\mathbf{F}(\hat{f})$ is just a disk bundle over $\mathbf{F}(\hat{f}_0)$, $\mathbf{C}(\hat{f})$ is just a disk bundle over $\mathbf{C}(\hat{f}_0)$. We may therefore extend the map \hat{f}_0 to a map \hat{f}

$$\begin{array}{ccc}
\mathbf{C}(\hat{f}) & \xrightarrow{\hat{f}} & \hat{\mathbf{B}} \\
\downarrow & & \downarrow \\
\mathbf{F}(\hat{f}) & \longrightarrow & \mathbf{B}_0
\end{array}$$

so that $ev^{+n}(\hat{f}) \circ s : \mathbf{F}(\hat{f}) \rightarrow \mathcal{M}(pt) \times \hat{\mathbf{B}}_{\mathbf{B}_0}^n$ is the identity inclusion of V . As this condition is G -equivariant and the original map \hat{f}_0 is G -invariant, we may construct our map \hat{f} to be G -invariant.

As \hat{f} is just the extension of \hat{f}_0 to a disk bundle, Lemma 4.4 of [15] implies that \hat{f} has universal tropical structure. Therefore \hat{f} satisfies condition 5 of Theorem 5.2.

By construction, $ev^{+n}(\hat{f}) \circ s$ is an embedding, and the derivative of $ev^{+n}(\hat{f})$ at $s(f)$ is an isomorphism. Therefore, by restricting \hat{f} to a smaller G -invariant neighborhood of f if necessary, $ev^{+n}(\hat{f})$ is an equidimensional embedding in a neighborhood of the image of s . In other words, \hat{f} satisfies condition 4 of Theorem 5.2.

Condition 3 of Theorem 5.2 is satisfied because s is G -equivariant. Condition 2 is satisfied because of item 5 from the construction of $\{s_i\}$.

We shall now verify Condition 6 of Theorem 5.2. We have already established that there are precisely $|G|$ intersections of $\lceil ev^{+n}(f) \rceil$ with $\lceil ev^{+n}(\hat{f}_0) \circ s \rceil$ corresponding to the $|G|$ maps of f into \hat{f}_0 . The corresponding intersections of $ev^{+n}(f)$ with the image of $ev^{+n}(\hat{f}) \circ s$ are transverse (and 0 dimensional). By restricting \hat{f} to a smaller G -equivariant neighborhood of \hat{f}_0 if necessary, we therefore get that there are precisely $|G|$ intersections of $\lceil ev^{+n}(f) \rceil$ with the closure of the image of

$[ev^{+n}(\hat{f}) \circ s]$, and that for any f' sufficiently close to f in \hat{f} , there are precisely $|G|$ intersections of $ev^{+n}(f')$ with $ev^{+n}(\hat{f}) \circ s$, that these intersections are transverse, and that there are no further intersections of $[ev^{+n}(f')]$ with the closure of $[ev^{+n}(\hat{f}) \circ s]$. By further reducing the size of \hat{f} , we may ensure that for all f' in \hat{f} , $ev^{+n}(f')$ intersects the closure of the image of $ev^{+n}(\hat{f}) \circ s$ exactly $|G|$ times. In other words, Condition 6 of Theorem 5.2 holds.

To verify Condition 1 of Theorem 5.2, we must verify that for all f' in \hat{f} , the action of G on the set of maps $f' \rightarrow f$ is free and transitive. As we have already shown that there are precisely $|G|$ intersections of $ev^{+n}(f')$ with $ev^{+n}(\hat{f}) \circ s$, there are at most $|G|$ maps $f' \rightarrow f$, and it remains to verify that the action of G on the set of these maps is free. This is easy, because the action of G does not fix the image of $[C(f)]$ in $[C(\hat{f})]$ under the inclusion $f \rightarrow \hat{f}$, so the action of G can not fix any curve in a G -equivariant neighborhood of f within \hat{f} . Therefore, Condition 1 of Theorem 5.2 will hold if we restrict \hat{f} to a small enough G -invariant open neighborhood of f .

We have now verified that $(\hat{f}/G, \{s_i\})$ satisfies all the conditions of Theorem 5.2, so $(\hat{f}/G, \{s_i\})$ is a core family of curves. \square

5.2. Shrinking open subsets of \mathcal{M}^{st} .

Lemma 5.10. *Suppose that the map $\mathcal{M}(\hat{\mathbf{B}}) \rightarrow \mathbf{B}_0$ is proper when \mathcal{M} is restricted to any connected component of $\mathcal{M}^{st}(\hat{\mathbf{B}})$.*

If \hat{f}/G is a core family for the open substack \mathcal{O} of $\mathcal{M}^{st}(\hat{\mathbf{B}})$, and f is a holomorphic curve in \hat{f} , there exists a continuous function $\rho : \mathcal{O} \rightarrow \mathbb{R}$ so that $\rho(f) = 1$, and so that any holomorphic curve in the closure of the set where $\rho > 0$ is contained in \mathcal{O} .

Proof:

The proof of this lemma is long, but roughly speaking, it is true because weak convergence of holomorphic curves to f , which may be detected by continuous functions on \mathcal{O} may be strengthened to imply $C^{\infty,1}$ convergence.

Choose a metric on $T_{vert}\hat{\mathbf{B}}$. Recall that Definition 5.1 tells us that given any family of curves $\hat{h} \in \mathcal{O}$, there is a canonical map

$$\begin{array}{ccc} \mathbf{C}(\hat{h}) & \xrightarrow{\Phi_{\hat{h}}} & \mathbf{C}(\hat{f})/G \\ \downarrow & & \downarrow \\ \mathbf{F}(\hat{h}) & \xrightarrow{\phi_{\hat{h}}} & \mathbf{F}(\hat{f})/G \end{array}$$

and a canonical section $\psi_{\hat{h}}$ of $(\hat{f} \circ \Phi_{\hat{h}})^*T_{vert}\hat{\mathbf{B}}$ so that h is $\hat{f} \circ \Phi_{\hat{h}}$ followed by exponentiation of $\psi_{\hat{h}}$.

Choose a G -invariant $C^{\infty,1}$ function $r_0 : \mathbf{F}(\hat{f}) \rightarrow [0, \infty)$ so that any sequence of points in $\mathbf{F}(\hat{f})$ for which r_0 converges to 0 converge to $f \in \mathbf{F}(\hat{f})$ or some G translate of f . For any family of curves \hat{h} in \mathcal{O} define

$$r : \mathbf{F}(\hat{h}) \rightarrow [0, \infty)$$

as

$$r(h) := \sup_{\mathbf{C}(h) \subset \mathbf{C}(\hat{h})} |\psi_{\hat{h}}| + r_0 \circ \phi_h$$

For each family \hat{h} in \mathcal{O} , $r : \mathbf{F}(\hat{h}) \rightarrow [0, \infty)$ is continuous. Given any map $\hat{g} \rightarrow \hat{h}$, the map $\mathbf{C}(\hat{g}) \rightarrow \mathbf{C}(\hat{h})$ pulls back $\phi_{\hat{h}}$ to $\phi_{\hat{g}}$, and $\psi_{\hat{h}}$ to $\psi_{\hat{g}}$. It follows that r is compatible with maps between families in \mathcal{O} and defines a continuous function

$$r : \mathcal{O} \rightarrow [0, \infty)$$

Claim 5.11. *There exists an $\epsilon > 0$ so that any holomorphic curve in the closure within \mathcal{M}^{st} of the set where $r < \epsilon$ must be contained in \mathcal{O} .*

To prove Claim 5.11, suppose to the contrary that there exists a sequence of holomorphic curves $\{h_i\}$ not in \mathcal{O} so that for all ϵ , $\{h_i\}$ is eventually contained in the closure of the set where $r < \epsilon$. It follows that the image of h_i in \mathbf{B}_0 converges, and $\{h_i\}$ is eventually contained in a connected component of \mathcal{M}^{st} . Our assumption on the properness of the map $\mathcal{M} \rightarrow \mathbf{B}_0$ then implies that some subsequence of $\{h_i\}$ converges in $C^{\infty, \perp}$ to a stable holomorphic curve h .

As \mathcal{O} is open, our holomorphic curve h must not be in \mathcal{O} . On the other hand, for all ϵ , h is in the closure of the subset of \mathcal{O} where $r < \epsilon$. We shall achieve a contradiction by showing that this implies that h and f have the same smooth part, which implies that h is actually in \mathcal{O} . This contradiction will take over two pages to achieve, so be patient.

There is a sequence of curves $\{f_i\}$ in \mathcal{O} which converge in $C^{\infty, \perp}$ to h , and so that $r(f_i)$ converges to 0. Lemma 2.7 implies that by passing to a subsequence, we may assume that f_i converge to h within some $C^{\infty, \perp}$ family \hat{h} containing h .

As $r_0 \circ \phi_{f_i}$ converges to 0, the images of $\mathbf{C}(f_i)$ in $\mathbf{C}(\hat{f})/G$ converge to $\mathbf{C}(f)$. By a judicious choice of resolution of the G -fold ambiguity of the map Φ_{f_i} , we may obtain inclusions $\mathbf{C}(f_i) \rightarrow \mathbf{C}(\hat{f})$ converging to $\mathbf{C}(f)$ which we shall again call Φ_{f_i} . Choose a metric on both $\mathbf{C}(\hat{h})$ and $\mathbf{C}(\hat{f})$. Consider the map Φ_{f_i} as mapping $\mathbf{C}(f_i) \subset \mathbf{C}(\hat{h})$ to $\mathbf{C}(f_i) \subset \mathbf{C}(\hat{f})$. We can use our metrics on $\mathbf{C}(\hat{h})$ and $\mathbf{C}(\hat{f})$ to measure the derivative of Φ_{f_i} .

Claim 5.12. *The derivatives of Φ_{f_i} and $\Phi_{f_i}^{-1}$ are uniformly bounded for all i .*

The proof of Claim 5.12 is a standard bubbling argument. Suppose that there was not a uniform derivative bound for Φ_{f_i} . Note that the injectivity radius of $\mathbf{C}(f_i)$ is uniformly bounded below in both $\mathbf{C}(\hat{h})$ and $\mathbf{C}(\hat{f})$ because the curves $\mathbf{C}(f_i)$ converge to $\mathbf{C}(h)$ and $\mathbf{C}(f)$ respectively. Therefore, there must be a sequence of injective, holomorphic maps x_i from the complex disk of size $3R_i$ to X so that

- $R_i \rightarrow \infty$
- the derivative at 0 of $\Phi_{f_i} \circ x_i$ has size equal to 1,
- the diameter of the image of x_i converges to 0

If there is a point inside the disk of radius R_i at which the derivative of $\Psi_{\hat{f}_i} \circ x_i$ has size greater than 2, we may recenter x_i at this point and rescale to obtain a replacement map obeying the above conditions. If the resulting rescaled map has a point in the disk of radius R_i with derivative greater than 2, repeat this procedure until the resulting rescaled map x'_i has the derivative of $\Phi_{f_i} \circ x'_i$ bounded by 2 within the disk of radius R_i . (This process must terminate because the derivative of $\Phi_{f_i} \circ x_i$ is bounded within the closed disk of radius $2R_i$.)

As these maps $\Phi_{f_i} \circ x'_i$ are holomorphic, the uniform bound on their derivative on the disk of radius R_i implies uniform bounds on their higher derivatives on the disk of radius $R_i - 1$. The Arzela-Ascoli theorem and the fact that the images of $\Phi_{f_i} \circ x'_i$

converge to $\mathbf{C}(f)$ implies that the sequence of maps $\Phi_{f_i} \circ x'_i$ must have a subsequence which converges on compact subsets to a holomorphic map $l : \mathbb{C} \rightarrow \mathbf{C}(f) \subset \mathbf{C}(\hat{f})$. As the derivative of $\Phi_{f_i} \circ x'_i$ has size 1 at 0, l is not constant. Given any family of taming forms on $\mathbf{C}(\hat{f})$, l also has bounded energy because $\Phi_{f_i} \circ x'_i$ is injective, and therefore has a uniform energy bound independent of i . There is also a uniform bound on the area of the image of $\Phi_{f_i} \circ x'_i$ contained within any disk of radius 1, therefore there is a uniform bound on the area of l contained within any disk of radius 1.

Note that as $r(f_i)$ converges to 0, the distance between $\hat{h} \circ x'_i$ and $\hat{f} \circ \Phi_{f_i} \circ x'_i$ converges to 0. As the diameter the image of x_i converges to 0, the diameter of $\hat{h} \circ x'_i$ converges to zero. It follows that the image of $f \circ l$ is a single point. We have that l is a nonconstant holomorphic map $\mathbb{C} \rightarrow \mathbf{C}(f)$ with bounded energy and local area so that $f \circ l$ is constant. No such map exists because $\lceil f \rceil$ is stable. We have achieved the required contradiction which proves that Φ_{f_i} has a uniform derivative bound independent of i . The argument to prove that $\Phi_{f_i}^{-1}$ has a uniform derivative bound is analogous. Simply switch the roles of \hat{f} and \hat{h} , and use $\Phi_{f_i}^{-1}$ instead of Φ_{f_i} . This completes the proof of Claim 5.12.

Claim 5.13. *By passing to a subsequence of $\{f_i\}$, we may assume that the following holds: There exists a bijective holomorphic map*

$$L : \lceil \mathbf{C}(h) \rceil \rightarrow \lceil \mathbf{C}(f) \rceil$$

so that given any point $a \in \lceil \mathbf{C}(h) \rceil$, then $b \in \lceil \mathbf{C}(f) \rceil$ is equal to $L(a)$ if and only if (a, b) satisfies the following property:

Common limit property: *For every sequence of points $a_i \in \mathbf{C}(f_i)$ converging to a in $\lceil \mathbf{C}(\hat{h}) \rceil$, $\Phi_{f_i}(a_i)$ converges to b in $\lceil \mathbf{C}(\hat{f}) \rceil$.*

As the maps Φ_{f_i} and $\Phi_{f_i}^{-1}$ are holomorphic, the uniform bound on their derivatives implies a uniform bound on their higher derivatives. Choose an exhaustion of $\mathbf{C}(h)$ minus its edges by compact subsets X_i . The fact that f_i converges within \hat{h} to h implies that by passing $\{f_i\}$ to a subsequence, we may assume that there exist maps

$$y_i : X_i \rightarrow \mathbf{C}(f_i)$$

which converge in C_c^∞ within $\lceil \mathbf{C}(\hat{h}) \rceil$ to the identity inclusion of $\mathbf{C}(h)$ minus its edges. Then the Arzela-Ascoli theorem implies that some subsequence of $\Phi_{f_i} \circ y_i$ converges in C_c^∞ to a smooth holomorphic map from $\mathbf{C}(h)$ minus its edges to $\lceil \mathbf{C}(f) \rceil \subset \lceil \mathbf{C}(\hat{f}) \rceil$. Define L restricted to $\mathbf{C}(h)$ minus its edges to be this map. Note that for any a in $\mathbf{C}(f)$ minus its edges, $(a, L(a))$ satisfies the common limit property.

Similarly, by passing to a subsequence of $\{f_i\}$, we may assume that for any b in $\mathbf{C}(f)$ minus its edges, there exists a unique point $a \in \lceil \mathbf{C} \rceil$ so that (a, b) satisfies the common intersection property.

If a is a node or puncture of $\lceil \mathbf{C}(h) \rceil$, there exists a sequence of points $a_i \in \mathbf{C}(f_i)$ which converge to a . By passing to a subsequence, we may assume that $\Phi_{f_i}(a_i)$ converges to a point b in $\mathbf{C}(f)$. Given any $\epsilon > 0$, any small enough neighborhood of a will intersect $\mathbf{C}(f_i)$ for i large enough within an (exploded) annulus which has image under $\lceil \hat{h} \rceil$ of diameter less than ϵ . Any sequence $a_i \in \mathbf{C}(f_i)$ which converges to a must eventually be contained within this neighborhood, and be a distance R_i from its boundary where $R_i \rightarrow \infty$. Choose ϵ small enough that the stability of $\lceil f \rceil$ implies that there are no unstable components of $\lceil \mathbf{C}(f) \rceil$ which have image under $\lceil f \rceil$ with diameter as small as ϵ . Then all points $b \in \mathbf{C}(f)$ which are the limit of

$\Phi_{f_i}(a_i)$ for some sequence $a_i \rightarrow a$ are contained inside some exploded annulus within $\mathbf{C}(f)$ with arbitrarily small image under $\lceil f \rceil$, and are an infinite distance from the boundaries of this exploded annulus. As $\lceil f \rceil$ is stable, any \mathbb{C}^* contained within $\mathbf{C}(f)$ with sufficiently small image under $\lceil f \rceil$ must have image a single point in $\lceil \mathbf{C}(f) \rceil$. It follows that all such $b \in \mathbf{C}(f)$ must have the same image in $\lceil \mathbf{C}(f) \rceil$. In other words, there is unique point $b \in \lceil \mathbf{C}(f) \rceil$ so that (a, b) satisfies the common limit property.

As there are only a finite number of nodes or punctures of $\lceil \mathbf{C}(h) \rceil$, by passing to a subsequence we may achieve that for all $a \in \lceil \mathbf{C}(h) \rceil$, there exists a unique $b \in \lceil \mathbf{C}(f) \rceil$ so that (a, b) satisfies the common limit property. Define $L(a) = b$. Similarly, we may assume that for all $b \in \lceil \mathbf{C}(f) \rceil$, there exists a unique a so that (a, b) satisfies the common limit property. It follows that L is a bijection. Clearly, L preserves limits, so L is continuous. As we already know that L is holomorphic on a dense subset, we have that L is a holomorphic bijection. This completes the proof of Claim 5.13.

Note that any holomorphic bijection must send nodes to nodes. The common limit property also implies that L sends punctures to punctures because for any puncture in $\lceil \mathbf{C}(h) \rceil$, there is a sequence of punctures in $\lceil \mathbf{C}(f_i) \rceil$ converging to it. The common limit property and the fact that $|\psi_{f_i}|$ converges to 0 implies that L is compatible with the maps $\lceil f \rceil$ and $\lceil h \rceil$:

$$\lceil f \rceil \circ L = \lceil h \rceil$$

Therefore, $\lceil f \rceil = \lceil h \rceil$. The convergence of f_i to h and $\hat{f} \circ \Phi_{f_i}$ to f imply that the derivative of each edge of \underline{f} coincides with that of \underline{h} . Therefore f and h are topologically indistinguishable within \mathcal{M}^{st} , so h must be in any open neighborhood of f . This is the desired contradiction which proves Claim 5.11.

We now have that there exists some ϵ so that all holomorphic curves in the closure of $\{r > \epsilon\}$ are contained in \mathcal{O} . To complete the proof of our lemma, all we need to do is compose r with a cut off function which is equal to 1 at 0 and which vanishes outside of an ϵ -neighborhood of 0. The resulting continuous function $\rho : \mathcal{O} \rightarrow [0, 1]$ is 1 at f and has the property that any holomorphic curve in the closure of $\{\rho > 0\}$ is contained in \mathcal{O} . This completes the proof of Lemma 5.10. \square

6. LOCALLY REPRESENTING THE MODULI OF STACK SOLUTIONS TO $\bar{\partial}f \in V$

The goal of this section is to prove Theorem 6.6 which roughly states that for a simply generated subsheaf V of \mathcal{Y} which is transverse to $\bar{\partial}$ at a holomorphic curve f , the moduli stack of curves close to f with $\bar{\partial}$ contained in V is locally represented by the quotient of a family of curves \hat{f} by a finite group G of automorphisms. Proposition 6.1 below may be regarded as a way of locally representing the moduli stack of holomorphic which are parametrized by the domain of a particular family of holomorphic curves. This proposition is then used in Lemma 6.4 to prove that an arbitrary simply generated subsheaf V may be parametrized by a family of curves. With V written in this special form, the results of [13] may then be used to prove Theorem 6.6.

Proposition 6.1. *Given any family of holomorphic curves \hat{f} in $\hat{\mathbf{B}}$ containing a given curve f , there exists a $C^{\infty, \perp}$ family of curves \hat{f}' in $\hat{\mathbf{B}}$ satisfying the following properties:*

- (1) There is an inclusion $\iota : \hat{f} \longrightarrow \hat{f}'$ and map

$$\begin{array}{ccc} \mathbf{C}(\hat{f}') & \xrightarrow{\psi} & \mathbf{C}(\hat{f}) \\ \downarrow & & \downarrow \\ \mathbf{F}(\hat{f}') & \longrightarrow & \mathbf{F}(\hat{f}) \end{array}$$

which is a holomorphic isomorphism restricted to fibers so that $\mathbf{C}(\hat{f}')$ is isomorphic to a vector bundle over $\mathbf{C}(\hat{f})$ with projection ψ and zero section ι .

- (2) There exists an open neighborhood \mathcal{O} of (id, f) in the moduli stack of $C^{\infty,1}$ curves in $\mathbf{C}(\hat{f}) \times \hat{\mathbf{B}}$, so that given any family \hat{h} of holomorphic curves in $\hat{\mathbf{B}}$ with a map

$$\begin{array}{ccc} \mathbf{C}(\hat{h}) & \xrightarrow{\psi'} & \mathbf{C}(\hat{f}) \\ \downarrow & & \downarrow \\ \mathbf{F}(\hat{h}) & \longrightarrow & \mathbf{F}(\hat{f}) \end{array}$$

which is a holomorphic isomorphism restricted to fibers, so that (ψ', \hat{h}) is in \mathcal{O} , then there exists a unique map $\hat{h} \longrightarrow \hat{f}'$ with the property that it pulls ψ back to ψ' .

$$\begin{array}{ccc} \mathbf{C}(\hat{h}) & \xrightarrow{\hat{h}} & \hat{\mathbf{B}} \\ \downarrow \psi' & \searrow \exists! & \uparrow \hat{f}' \\ \mathbf{C}(\hat{f}) & \xleftarrow{\psi} & \mathbf{C}(\hat{f}') \end{array}$$

Suppose that $\hat{\mathbf{B}}$ has a group G_0 of automorphisms and there is a group $G_0 \times G$ of automorphisms of $\mathbf{C}(\hat{f}) \longrightarrow \mathbf{F}(\hat{f})$ so that \hat{f} is G -invariant and G_0 -equivariant. Then \hat{f}' may be constructed so that $\mathbf{C}(\hat{f}') \longrightarrow \mathbf{F}(\hat{f}')$ has a group $G_0 \times G$ of automorphisms, ψ and ι are $G_0 \times G$ -equivariant, and \hat{f}' is G -invariant and G_0 -equivariant.

Remark 6.2. The intersection of $\bar{\partial}\hat{f}'$ with 0 should be regarded as representing the moduli space of holomorphic curves close to f which are parametrized by $\mathbf{C}(\hat{f}) \longrightarrow \mathbf{F}(\hat{f})$. In the case when $G_0 \times G$ is nontrivial,

$$\{f' \in \hat{f}' \text{ so that } \bar{\partial}f' = 0\} / G_0 \times G$$

should be regarded as representing the moduli stack of holomorphic curves in $\hat{\mathbf{B}}/G_0$ parametrized by $\mathbf{C}(\hat{f})/(G_0 \times G)$. The point of Proposition 6.1 is that this moduli stack is locally represented as a subset of a finite dimensional $C^{\infty,1}$ family quotiented out by a finite group action.

Proof:

We shall prove the equivariant case. Using a G_0 -invariant, smooth, J preserving connection on $f^*T_{\text{vert}}\hat{\mathbf{B}}$, we may construct a $G_0 \times G$ -invariant trivialization (F, ϕ) to associate to \hat{f} . More precisely, the vector bundle $f^*T_{\text{vert}}\hat{\mathbf{B}}$ has a $G \times G_0$ action. The map $F : f^*T_{\text{vert}}\hat{\mathbf{B}} \longrightarrow \hat{\mathbf{B}}$ given by exponentiating using our invariant connection (and reparametrizing in a $G_0 \times G$ -equivariant way to ensure injectivity of TF restricted to any vertical tangent space) is G -invariant and G_0 -equivariant, and the vector bundle map $\phi : F^*T_{\text{vert}}\hat{\mathbf{B}} \longrightarrow \hat{f}^*T_{\text{vert}}\hat{\mathbf{B}}$ defined using parallel transport along a straight line homotopy using our connection is $G_0 \times G$ -equivariant.

Using such a trivialization, $\bar{\partial} : X^{\infty,1}(\hat{f}) \longrightarrow \mathcal{Y}(\hat{f})$ is $G_0 \times G$ -equivariant.

Given any family of curves \hat{h} in $\hat{\mathbf{B}}$ with a map

$$\begin{array}{ccc} \mathbf{C}(\hat{h}) & \xrightarrow{\psi'} & \mathbf{C}(\hat{f}) \\ \downarrow & & \downarrow \\ \mathbf{F}(\hat{h}) & \longrightarrow & \mathbf{F}(\hat{f}) \end{array}$$

which is a holomorphic isomorphism restricted to fibers, so long as $\hat{f} \circ \psi'$ is close enough to \hat{h} , the trivialization associated with \hat{f} allows us to uniquely factor \hat{h} as F composed with a map

$$(6) \quad \nu_{\hat{h}} : \mathbf{C}(\hat{h}) \longrightarrow f^*T_{vert}\hat{\mathbf{B}}$$

so that the following diagram commutes.

$$\begin{array}{ccccc} & & \hat{h} & & \\ & \swarrow & & \searrow & \\ \mathbf{C}(\hat{h}) & \xrightarrow{\nu_{\hat{h}}} & f^*T_{vert}\hat{\mathbf{B}} & \xrightarrow{F} & \hat{\mathbf{B}} \\ & \searrow \psi' & \downarrow & & \\ & & \mathbf{C}(\hat{f}) & & \end{array}$$

Choose some $G_0 \times G$ -invariant collection of marked point sections $\{s_i\}$ of $\mathbf{C}(\hat{f}) \longrightarrow \mathbf{F}(\hat{f})$ so that $D\bar{\partial}$ restricted to sections of $f^*T_{vert}\hat{\mathbf{B}}$ which vanish at these extra marked points is injective. The vector bundle

$$\begin{array}{c} \oplus_i (s_i \circ \hat{f})^*T_{vert}\hat{\mathbf{B}} \\ \downarrow \\ \mathbf{F}(\hat{f}) \end{array}$$

has a natural $G_0 \times G$ action. Construct a family of curves \hat{f}'_0 with domain given by the following pullback

$$(7) \quad \begin{array}{ccc} \mathbf{C}(\hat{f}'_0) & \longrightarrow & \mathbf{C}(\hat{f}) \\ \downarrow & & \downarrow \\ \oplus_i (s_i \circ \hat{f})^*T_{vert}\hat{\mathbf{B}} & \longrightarrow & \mathbf{F}(\hat{f}) \end{array}$$

Note that $\mathbf{C}(\hat{f}'_0)$ has a natural $G_0 \times G$ action so that the above diagram is $G_0 \times G$ -equivariant. We may pullback the sections s_i to give a $G_0 \times G$ -invariant collection of sections $\{s'_i\}$ of $\mathbf{C}(\hat{f}'_0) \longrightarrow \mathbf{F}(\hat{f}'_0)$.

$$\begin{array}{ccc} \mathbf{C}(\hat{f}'_0) & \longrightarrow & \mathbf{C}(\hat{f}) \\ \downarrow \uparrow s'_i & & \downarrow \uparrow s_i \\ \oplus_i (s_i \circ \hat{f})^*T_{vert}\hat{\mathbf{B}} & \longrightarrow & \mathbf{F}(\hat{f}) \end{array}$$

Construct \hat{f}'_0 so that \hat{f}'_0 factors as a $G_0 \times G$ -equivariant map ν to $\hat{f}^*T_{vert}\hat{\mathbf{B}}$ followed by F , so that the 5 inner loops in the following diagram commute,

$$\begin{array}{ccccc}
 & & \hat{f}'_0 & & \\
 & \nearrow & & \searrow & \\
 \mathbf{C}(\hat{f}'_0) & \xrightarrow{\nu} & \hat{f}^*T_{vert}\hat{\mathbf{B}} & \xrightarrow{F} & \hat{\mathbf{B}} \\
 \downarrow s'_i & \searrow & \uparrow & \nearrow & \downarrow \\
 \mathbf{F}(\hat{f}'_0) & & \mathbf{C}(\hat{f}) & & \mathbf{G} \\
 & \nearrow & \downarrow s_i & \searrow & \\
 & & \mathbf{F}(\hat{f}) & &
 \end{array}$$

and so that if $\mathbf{C}(\hat{f}'_0)$ is considered as a vector bundle over $\mathbf{C}(\hat{f})$, ν is a map of vector bundles, and

$$\nu \circ s'_i : \oplus (s_i \circ \hat{f})^*T_{vert}\hat{\mathbf{B}} \longrightarrow (s_i \circ \hat{f})^*T_{vert}\hat{\mathbf{B}}$$

is projection onto the i th factor.

Evaluating the section $\nu_{\hat{h}}$ from equation (6) at the image of each section s_i

$$\mathbf{C}(\hat{h}) \longrightarrow \mathbf{F}(\hat{h}) \xrightarrow{(\psi')^*s_i} \mathbf{C}(\hat{h}) \xrightarrow{\nu_{\hat{h}}} (s_i \circ \hat{f})^*T_{vert}\hat{\mathbf{B}}$$

defines a map

$$\mathbf{C}(\hat{h}) \longrightarrow \oplus_i (s_i \circ \hat{f})^*T_{vert}\hat{\mathbf{B}}$$

so that the following diagram commutes:

$$\begin{array}{ccc}
 \mathbf{C}(\hat{h}) & \xrightarrow{\psi'} & \mathbf{C}(\hat{f}) \\
 \downarrow & & \downarrow \\
 \oplus_i (s_i \circ \hat{f})^*T_{vert}\hat{\mathbf{B}} & \longrightarrow & \mathbf{F}(\hat{f})
 \end{array}$$

As $\mathbf{C}(\hat{f}'_0)$ is defined by the pullback diagram (7) there is an induced map

$$\begin{array}{ccc}
 \mathbf{C}(\hat{h}) & \xrightarrow{\psi''} & \mathbf{C}(\hat{f}'_0) \\
 \downarrow & & \downarrow \\
 \mathbf{F}(\hat{h}) & \longrightarrow & \oplus_i (s_i \circ \hat{f})^*T_{vert}\hat{\mathbf{B}}
 \end{array}$$

which is a holomorphic isomorphism on each fiber because ψ' factorizes as ψ'' composed with the map $\mathbf{C}(\hat{f}') \longrightarrow \mathbf{C}(\hat{f})$ which is a holomorphic isomorphism on each fiber. Roughly speaking, this lift of ψ' is determined by the condition that \hat{h} agrees with $f' \circ \psi''$ when restricted to the pullback under ψ' of the sections s_i .

We may now pullback our constructed trivialization or construct another trivialization using the same connection to give a $(G_0 \times G)$ -invariant trivialization to associate to our new family \hat{f}'_0 . Again, call this trivialization (F, ϕ) . As before we may use this trivialization to factorize \hat{h} as follows:

$$\begin{array}{ccccc}
 & & \hat{h} & & \\
 & \nearrow & & \searrow & \\
 \mathbf{C}(\hat{h}) & \xrightarrow{\nu'_{\hat{h}}} & (f'_0)^*T_{vert}\hat{\mathbf{B}} & \xrightarrow{F} & \hat{\mathbf{B}} \\
 & \searrow & \downarrow \psi'' & & \\
 & & \mathbf{C}(\hat{f}'_0) & &
 \end{array}$$

where ν'_h is uniquely determined by the conditions that it must vanish on the image of the sections s_i , the above diagram commutes, and $\hat{h} = F \circ \nu'_h$.

At $f \in \hat{f}_0$, $D\bar{\partial}$ is injective restricted to sections of $f^*T_{vert}\hat{\mathbf{B}}$ which vanish on marked points corresponding to the sections s_i . Theorem 3.8 implies that we may choose a $G_0 \times G$ -invariant, finite dimensional sub vectorbundle V of $\mathcal{Y}(\hat{f}_0)$ so that the pre-obstruction model (\hat{f}'_0, V) has $D\bar{\partial} : X^{\infty, \perp}(f) \rightarrow \mathcal{Y}(f)$ transverse to $V(f)$. We may then apply Theorem 3.6 to (\hat{f}'_0, V) to obtain a unique section ν of $(\hat{f}'_0)^*T_{vert}\hat{\mathbf{B}}$ defined near f and vanishing on the image of all s'_i so that $\bar{\partial}\nu$ is a section of V .

Claim 6.3. *There exists a neighborhood \mathcal{O} of (id, f) in $\mathcal{M}^{st}(\mathbf{C}(\hat{f}) \times \hat{\mathbf{B}})$ so that if \hat{h} is in \mathcal{O} , then ν'_h is the pullback of ν under the following diagram:*

$$\begin{array}{ccc} \mathbf{C}(\hat{h}) & \xrightarrow{\nu'_h} & (f'_0)^*T_{vert}\hat{\mathbf{B}} \xrightarrow{F} \hat{\mathbf{B}} \\ & \searrow \psi'' & \uparrow \nu \downarrow \\ & & \mathbf{C}(\hat{f}'_0) \end{array}$$

To prove Claim 6.3, we may choose any neighborhood \mathcal{O}' of (id, f) so that the construction of ν'_h makes sense for all \hat{h} in this neighborhood, then let \mathcal{O}' be \mathcal{O} minus all holomorphic curves h which do not satisfy Claim 6.3 or for which $D\bar{\partial} : X^{\infty, \perp}(h) \rightarrow \mathcal{Y}(h)$ is not transverse to $V(h)$.

We need only verify that \mathcal{O} is open. Suppose to the contrary that there was a sequence of holomorphic curves in $\mathcal{O}' \setminus \mathcal{O}$ which converged to a holomorphic curve h in \mathcal{O} . Lemma 2.7 implies that some subsequence h_i must be contained in a family \hat{h} in \mathcal{O}' so that h_i converges to h within \hat{h} . We may pull back our obstruction model to give an obstruction model on \hat{h} . Then Theorem 3.6 implies that $D\bar{\partial} : X^{\infty, \perp}(h_i) \rightarrow \mathcal{Y}(h_i)$ is transverse to $V(h)$ for i large enough, and that there exists a unique section ν_0 in $X^{\infty, \perp}(\psi'' \circ \hat{f}_0)$ defined on a neighborhood of h which is close to ν'_h , and for which $\bar{\partial}F(\nu_0) \in V$. As $\nu \circ \psi''$ satisfies these conditions,

$$\nu_0 = \nu \circ \psi''$$

As $0 = \bar{\partial}(h_i)$, the uniqueness part of Theorem 3.6 implies that ν_0 also coincides with ν'_h when restricted to h_i for i large enough. Therefore, h_i must actually be inside \mathcal{O} for i large enough. This contradiction completes the proof of Claim 6.3.

Claim 6.3 implies that for \hat{h} in \mathcal{O} ,

$$\hat{h} = F(\nu) \circ \psi''$$

Letting $\hat{f}' := F(\nu)$, our map ψ'' therefore gives us the required unique map $\hat{h} \rightarrow \hat{f}'$ so that the following diagram commutes

$$\begin{array}{ccc} \mathbf{C}(\hat{h}) & \xrightarrow{\hat{h}} & \hat{\mathbf{B}} \\ \downarrow \psi' & \searrow \psi'' & \uparrow \hat{f}' \\ \mathbf{C}(\hat{f}) & \xleftarrow{\psi} & \mathbf{C}(\hat{f}') \end{array}$$

Because the map $\bar{\partial} : X^{\infty, \perp}(\hat{f}'_0) \rightarrow \mathcal{Y}(\hat{f}'_0)$ is defined using our $G_0 \times G$ -invariant trivialization, it is $G_0 \times G$ -equivariant and V is $G_0 \times G$ -invariant, the unique solution ν to $\bar{\partial}\nu \in V$ must also be $G_0 \times G$ -equivariant. As the map F from our trivialization

is G_0 -equivariant and G -invariant, our family \hat{f}' is G_0 -equivariant and G -invariant as required. \square

We wish to study the moduli stack of solutions to $\bar{\partial}\hat{f} \in V(\hat{f})$ where V is a simply generated subsheaf of \mathcal{Y} in the sense of Definition 2.20. The next lemma shows that any such V may locally be parametrized by a family of curves \hat{f}_1/G which may be regarded as a vector bundle over a core family \hat{f}_0/G . We shall use this result to apply the analysis from [13].

Lemma 6.4. *If V is a n -dimensional simply generated subsheaf of \mathcal{Y} defined on a neighborhood of a holomorphic curve f with at least one smooth component, then there exists a neighborhood \mathcal{O} of f in \mathcal{M}^{st} , a core family $(\hat{f}_0/G, \{s_i\})$ for \mathcal{O} and a G -invariant family of curves, \hat{f}_1 in \mathcal{O} together with a fiberwise holomorphic map*

$$\begin{array}{ccc} \mathcal{O}^{+1} & \longrightarrow & \mathbf{C}(\hat{f}_1)/G \\ \downarrow & & \downarrow \\ \mathcal{O} & \longrightarrow & \mathbf{F}(\hat{f}_1)/G \end{array}$$

so that

- (1) *There exists locally free, n -dimensional, G -invariant, sheaf V_1 of $C^{\infty,1}(\mathbf{F}(\hat{f}_1))$ -modules on $\mathbf{F}(\hat{f}_1)$ which is a subsheaf of $\Gamma^{(0,1)}(T_{vert}^* \mathbf{C}(\hat{f}_1) \otimes T_{vert} \hat{\mathbf{B}})$ and which pulls back to give V on \mathcal{O} . (See Definition 2.19 for the definition of V_1 pulling back to give V .)*
- (2) *\hat{f}_0 is a G -invariant sub family of \hat{f}_1 , and there is a G -equivariant projection*

$$\hat{f}_1 \longrightarrow \hat{f}_0$$

which is the identity on \hat{f}_0 so that $\mathbf{C}(\hat{f}_1)$ is isomorphic to a vector bundle over $\mathbf{C}(\hat{f}_0)$ with the given projection and zero section, and so that the core family map to $\mathbf{C}(\hat{f}_0)/G$ factorizes as

$$\begin{array}{ccccc} \mathcal{O}^{+1} & \longrightarrow & \mathbf{C}(\hat{f}_1)/G & \longrightarrow & \mathbf{C}(\hat{f}_0)/G \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{O} & \longrightarrow & \mathbf{F}(\hat{f}_1)/G & \longrightarrow & \mathbf{F}(\hat{f}_0)/G \end{array}$$

- (3) *For $t \in [0, 1]$, $((1-t)D\bar{\partial} + tD\bar{\partial}^C)^{-1}(V(f))$ does not contain any nonzero sections of $f^*T_{vert} \hat{\mathbf{B}}$ which vanish at the image of the marked point sections $\{s_i\}$.*

Proof:

Proposition 5.8 implies that there is some $C^{\infty,1}$ core family $(\hat{f}/G_1, \{s_i\})$ containing f . As indicated by Proposition 5.8 and Theorem 3.8, we may construct \hat{f} with enough marked point sections s_i that $((1-t)D\bar{\partial} + tD\bar{\partial}^C)^{-1}(V(f))$ contains no nonzero sections of $f^*T_{vert} \hat{\mathbf{B}}$ which vanish on the images of the s_i .

From the definition of a core family, there exists an open neighborhood \mathcal{O} of \hat{f} with a fiberwise holomorphic map

$$\begin{array}{ccc} \mathcal{O}^{+1} & \longrightarrow & \mathbf{C}(\hat{f})/G_1 \\ \downarrow & & \downarrow \\ \mathcal{O} & \longrightarrow & \mathbf{F}(\hat{f})/G_1 \end{array}$$

in the sense of Definition 2.5 on page 7. In other words, given any family of curves \hat{h} in \mathcal{O} , there is a G_1 -fold cover \hat{h}' of \hat{h} with a G_1 -equivariant map

$$\begin{array}{ccc} \mathbf{C}(\hat{h}') & \longrightarrow & \mathbf{C}(\hat{f}) \\ \downarrow & & \downarrow \\ \mathbf{F}(\hat{h}') & \longrightarrow & \mathbf{F}(\hat{f}) \end{array}$$

which is fiberwise holomorphic so that given any map $\hat{g} \rightarrow \hat{h}$, the following diagram commutes and is G_1 -equivariant

$$\begin{array}{ccc} \hat{g}' & \longrightarrow & \hat{h}' \\ \downarrow & & \downarrow \\ \hat{g} & \longrightarrow & \hat{h} \end{array}$$

and $\mathbf{C}(\hat{g}') \rightarrow \mathbf{C}(\hat{f})$ factors as

$$\mathbf{C}(\hat{g}') \longrightarrow \mathbf{C}(\hat{h}') \longrightarrow \mathbf{C}(\hat{f})$$

We may choose \mathcal{O} small enough that the maps $\mathbf{C}(\hat{h}') \rightarrow \mathbf{C}(\hat{f})$ are fiberwise isomorphisms.

The definition of a simply generated subsheaf V of \mathcal{Y} , Definition 2.20 on page 18, implies that if \mathcal{O} is small enough, there is also a fiberwise holomorphic map

$$\begin{array}{ccc} \mathcal{O}^{+1} & \longrightarrow & \hat{\mathbf{A}}/G_0 \\ \downarrow & & \downarrow \\ \mathcal{O} & \longrightarrow & \mathbf{X}/G_0 \end{array}$$

and sections v_1, \dots, v_n of $\Gamma^{(0,1)}(T_{vert}^* \hat{\mathbf{A}} \otimes T_{vert} \hat{\mathbf{B}})$ which pull back to generate V .

In particular, there is a G_0 -fold cover \hat{f}' of \hat{f} with a G_0 -equivariant, fiberwise holomorphic map

$$\hat{r} : \mathbf{C}(\hat{f}') \rightarrow \hat{\mathbf{A}}$$

Definition 2.5 implies that the action of G_1 on \hat{f} lifts to an action of G_1 on $\mathbf{C}(\hat{f}') = \mathbf{C}(\hat{r})$, so that \hat{r} is G_1 -invariant and G_0 -equivariant.

Claim 6.5. *The group of automorphisms of $\mathbf{C}(\hat{f}')$ generated by G_0 and G_1 is $G_0 \times G_1$.*

Let G be the group of automorphisms generated by G_0 and G_1 . Definition 2.5 implies that the action of G_1 on $\mathbf{C}(\hat{f}')$ must commute with the action of G_0 because each element of G_1 acts by a G_0 -equivariant map. Therefore, G must be a subgroup of $G_0 \times G_1$. The map $\hat{f}' \rightarrow \hat{f}$ is G_1 -equivariant and G_0 -invariant, and the action of G_1 on $\mathbf{C}(\hat{f})$ is effective. Therefore, there is a surjective homomorphism $G \rightarrow G_1$

with G_0 in the kernel. Therefore, G must be equal to $G_0 \times G_1$ and Claim 6.5 is proved.

The map to our core family \hat{f} specifies a G_1 -fold cover \hat{h}' of \hat{h} . The map $\mathcal{O}^{+1} \rightarrow \mathbf{A}/G_0$ specifies a G_0 -fold cover \hat{h}'' of \hat{h}' . We shall now verify that the cover $\hat{h}'' \rightarrow \hat{h}'$ may be regarded as the pullback of the G_0 -fold cover $\hat{f}' \rightarrow \hat{f}$.

The definition of a core family implies that there is a homotopy \hat{h}'_t of the G_1 -fold cover of the family \hat{h} to some family \hat{h}'_0 which maps to \hat{f} so that the maps

$$\begin{array}{ccc} \mathbf{C}(\hat{h}'_t) & \longrightarrow & \mathbf{C}(\hat{f}) \\ \downarrow & & \downarrow \\ \mathbf{F}(\hat{h}'_t) & \longrightarrow & \mathbf{F}(\hat{f}) \end{array}$$

do not depend on t .

The map $\mathcal{O}^{+1} \rightarrow \mathbf{A}/G_0$ specifies a G_0 -fold cover \hat{h}''_t of \hat{h}'_t . Restricted to $t = 0$, this G_0 -fold cover must be the pullback of the G_0 -fold cover \hat{f}' of \hat{f}

$$\begin{array}{ccc} \hat{h}''_0 & \longrightarrow & \hat{f}' \\ \downarrow & & \downarrow \\ \hat{h}'_0 & \longrightarrow & \hat{f} \end{array}$$

Therefore, the G_0 -fold cover $\mathbf{C}(\hat{h}'')$ of $\mathbf{C}(\hat{h}')$ must correspondingly be the pullback of the G_0 -fold cover $\mathbf{C}(\hat{f}')$ of $\mathbf{C}(\hat{f})$.

$$\begin{array}{ccc} \mathbf{C}(\hat{h}'') & \longrightarrow & \mathbf{C}(\hat{f}') \\ \downarrow & & \downarrow \\ \mathbf{C}(\hat{h}') & \longrightarrow & \mathbf{C}(\hat{f}) \end{array}$$

To summarize, we now have a $G_0 \times G_1$ -invariant family \hat{f}' , a G_0 -equivariant, G_1 -invariant, fiberwise holomorphic map $\hat{r} : \mathbf{C}(\hat{f}') \rightarrow \hat{\mathbf{A}}$, and a fiberwise holomorphic map

$$\begin{array}{ccc} \mathcal{O}^{+1} & \longrightarrow & (\mathbf{C}(\hat{f}') \times \hat{\mathbf{A}})/G_0 \times G_1 \\ \downarrow & & \downarrow \\ \mathcal{O} & \longrightarrow & (\mathbf{F}(\hat{f}') \times \mathbf{X})/G_0 \times G_1 \end{array}$$

Use Proposition 6.1 to extend \hat{r} to a family of curves \hat{r}' in $\hat{\mathbf{A}}$ with a map $\psi : \mathbf{C}(\hat{r}') \rightarrow \mathbf{C}(\hat{r})$ and with the property that given any family of curves \hat{h} close enough to \hat{f} , the map

$$\begin{array}{ccc} \mathbf{C}(\hat{h}'') & \xrightarrow{\psi'} & \mathbf{C}(\hat{f}') = \mathbf{C}(\hat{r}) \\ \downarrow & & \downarrow \\ \mathbf{F}(\hat{h}'') & \longrightarrow & \mathbf{F}(\hat{f}') = \mathbf{C}(\hat{r}) \end{array}$$

together with the fiberwise holomorphic map $\hat{r}_{\hat{h}} : \mathbf{C}(\hat{h}'') \rightarrow \hat{\mathbf{A}}$ give a unique map $\psi_1 : \hat{r}_{\hat{h}} \rightarrow \hat{r}'$ with the property that it pulls ψ back to ψ' .

$$(8) \quad \begin{array}{ccc} \mathbf{C}(\hat{h}'') & \xrightarrow{\hat{r}_{\hat{h}}} & \hat{\mathbf{A}} \\ \downarrow \psi' & \searrow \psi_1 & \uparrow \hat{r}' \\ \mathbf{C}(\hat{r}) & \xleftarrow{\psi} & \mathbf{C}(\hat{r}') \end{array}$$

Assume we have chosen \mathcal{O} small enough so that the above holds for any family of curves \hat{h} in \mathcal{O} .

The uniqueness property from Proposition 6.1 together with the observation that $\hat{r}_{\hat{h}}$ and \hat{r}' are G_1 -invariant and G_0 -equivariant, and ψ' and ψ are $G_0 \times G_1$ -equivariant implies that this map $\psi_1 : \mathbf{C}(\hat{h}'') \rightarrow \mathbf{C}(\hat{r}')$ is $G_0 \times G_1$ -equivariant. Similarly, this uniqueness property and the fact that ψ' and \hat{r} are fiberwise holomorphic maps from \mathcal{O}^{+1} implies that the above defines a fiberwise holomorphic map $\psi_1 : \mathcal{O}^{+1} \rightarrow \mathbf{C}(\hat{r}')/G_0 \times G_1$.

Let G be $G_0 \times G_1$, \hat{f}_0 be \hat{f}' , and the sections $\{s_i\}$ of $\mathbf{C}(\hat{f}_0) \rightarrow \mathbf{F}(\hat{f}_0)$ be the pullback of the corresponding sections $\{s_i\}$ of $\mathbf{C}(\hat{f}) \rightarrow \mathbf{F}(\hat{f})$. The core family $(\hat{f}_0/G, \{s_i\})$ is the core family referred to in the statement of this lemma. Define \hat{f}_1 to be the family of curves in $\hat{\mathbf{B}}$ with domain the same as \hat{r}' , pulled back from $\hat{f}' = \hat{f}_0$ via the map ψ .

$$\begin{array}{ccccc} & & \hat{f}_1 & & \\ & \curvearrowright & & \curvearrowleft & \\ \mathbf{C}(\hat{f}_1) = \mathbf{C}(\hat{r}') & \xrightarrow{\psi} & \mathbf{C}(\hat{r}) = \mathbf{C}(\hat{f}') = \mathbf{C}(\hat{f}_0) & \xrightarrow{\hat{f}' = \hat{f}_0} & \hat{\mathbf{B}} \\ \downarrow & & \downarrow & & \\ \mathbf{F}(\hat{f}_1) & \longrightarrow & \mathbf{F}(\hat{f}_0) & & \end{array}$$

Note that as $\mathbf{C}(\hat{f}_1) := \mathbf{C}(\hat{r}')$ is constructed using Proposition 6.1, $\mathbf{C}(\hat{f}_1)$ is isomorphic to a vector bundle over $\mathbf{C}(\hat{f}_0)$, as required for this lemma.

We now have a G -invariant family \hat{f}_1 and a fiberwise holomorphic map

$$\begin{array}{ccc} \mathcal{O}^{+1} & \xrightarrow{\psi_1} & \mathbf{C}(\hat{f}_1)/G \\ \downarrow & & \downarrow \\ \mathcal{O} & \longrightarrow & \mathbf{F}(\hat{f}_1)/G \end{array}$$

which is a factor in the core family map to $\mathbf{C}(\hat{f}_0)/G$

$$\begin{array}{ccccc} & & \hat{f}_1 & & \\ & \curvearrowright & & \curvearrowleft & \\ \mathcal{O}^{+1} & \xrightarrow{\psi_1} & \mathbf{C}(\hat{f}_1)/G & \xrightarrow{\psi} & \mathbf{C}(\hat{f}_0)/G \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{O} & \longrightarrow & \mathbf{F}(\hat{f}_1)/G & \longrightarrow & \mathbf{F}(\hat{f}_0)/G \end{array}$$

The commutative diagram (8) implies that our fiberwise holomorphic map $\psi_1 : \mathcal{O}^{+1} \rightarrow \mathbf{C}(\hat{f}_1)/G$ may also be regarded as a factor in the map $\mathcal{O}^{+1} \rightarrow \mathbf{A}/G_0$ in the following sense: The map $\mathbf{C}(\hat{h}'') \rightarrow \mathbf{C}(\hat{f}_1)$ composed with \hat{r}' gives a G_1 -invariant, G_0 -equivariant map $r_{\hat{h}} : \mathbf{C}(\hat{h}'') \rightarrow \hat{\mathbf{A}}$. Quotienting the domain by the

G_1 action gives the original map from the G_0 -fold cover of $\mathbf{C}(\hat{h})$ to $\hat{\mathbf{A}}$, so the original map to $\hat{\mathbf{A}}/G_0$ may be thought of as being factorized as follows

$$\begin{array}{ccccc} \mathcal{O}^{+1} & \xrightarrow{\psi_1} & \mathbf{C}(\hat{f}_1)/G & \xrightarrow{\hat{r}'} & \hat{\mathbf{A}}/G_0 \\ \downarrow & \searrow & \downarrow & \searrow & \downarrow \\ \mathcal{O} & \longrightarrow & \mathbf{F}(\hat{f}_1)/G & \longrightarrow & \mathbf{X}/G_0 \end{array}$$

We shall now examine how \hat{f}_1 may be used to parametrize V . The map

$$(\hat{r}', \text{id}) : \mathbf{C}(\hat{f}_1) \times \hat{\mathbf{B}} \longrightarrow \hat{\mathbf{A}} \times \hat{\mathbf{B}}$$

may be used to pull back the sections v_1, \dots, v_n of $\Gamma^{(0,1)}(T_{\text{vert}}^* \hat{\mathbf{A}} \otimes T_{\text{vert}} \hat{\mathbf{B}})$ to G_1 -equivariant sections v'_1, \dots, v'_n of $\Gamma^{(0,1)}(T_{\text{vert}}^* \mathbf{C}(\hat{f}_1) \otimes T_{\text{vert}} \hat{\mathbf{B}})$. Given any family \hat{h} in \mathcal{O} , the map

$$(\psi_1, \hat{h}'') : \mathbf{C}(\hat{h}'') \longrightarrow \mathbf{C}(\hat{f}_1) \times \hat{\mathbf{B}}$$

may be used to pull back v'_1, \dots, v'_n to G_1 -equivariant sections v''_1, \dots, v''_n of $\mathcal{Y}(\hat{h}'')$, which diagram 8 implies are the same as the sections obtained by pulling back v_1, \dots, v_n under the map

$$(\hat{r}_h, \hat{h}'') : \mathbf{C}(\hat{h}'') \longrightarrow \hat{\mathbf{A}} \times \hat{\mathbf{B}}$$

As the quotient of \hat{r}_h under the G_1 action on \hat{h}'' gives the original map from a G_0 -fold cover of $\mathbf{C}(\hat{h})$ to $\hat{\mathbf{A}}$, these sections v''_1, \dots, v''_n of $\mathcal{Y}(\hat{h}'')$ are linearly independent at every curve in \hat{h}'' and generate $V(\hat{h}'')$.

In other words, the pullback of v'_1, \dots, v'_n generate V on \mathcal{O} . As the sheaf of $C^{\infty,1}(\mathbf{X})$ modules over \mathbf{X} generated by v_1, \dots, v_n is G_0 -invariant and the map \hat{r}' is G_1 -invariant and G_0 -equivariant, the sheaf of $C^{\infty,1}(\mathbf{F})$ modules over $\mathbf{F}(\hat{f}_1)$ generated by v'_1, \dots, v'_n is G -invariant. As the pullback of v'_1, \dots, v'_n to any curve in \mathcal{O} are linearly independent, restricted to a neighborhood of the image of \mathcal{O} in $\mathbf{F}(\hat{f}_1)$, the sheaf V_1 generated by v'_1, \dots, v'_n is free and n dimensional. Therefore, by restricting \hat{f}_1 to a G -invariant neighborhood of the image of \mathcal{O} , v'_1, \dots, v'_n generate an n -dimensional G -invariant vector bundle V_1 which pulls back to give V on \mathcal{O} . \square

Theorem 6.6. *If V is a simply generated subsheaf of \mathcal{Y} so that for some holomorphic curve f , $V(f)$ is transverse to $D\bar{\partial} : T_f \mathcal{M}^{st}(\hat{\mathbf{B}}) \longrightarrow \mathcal{Y}(f)$, then there exists an open neighborhood \mathcal{O} of f and a $C^{\infty,1}$ family of curves \hat{f} with automorphism group G so that \hat{f}/G represents the substack $\{\bar{\partial} \in V\} \subset \mathcal{O}$ consisting of curves g in \mathcal{O} with the property that $\bar{\partial}g \in V(g)$.*

The map

$$T_f \mathbf{F}(\hat{f}) \longrightarrow T_f \mathcal{M}^{st}(\hat{\mathbf{B}})$$

is injective and has image equal to $D\bar{\partial}^{-1}(V(f)) \subset \mathcal{M}^{st}(\hat{\mathbf{B}})$.

Proof:

We shall prove the easy case in which the domain of f is \mathbf{T} separately in Lemma 7.5. For the rest of this proof, assume that the domain of f has at least one smooth component, so we may use Lemma 6.4 to construct a core family $(\hat{f}_0/G, \{s_i\})$ for an open neighborhood \mathcal{O} of f and a G -invariant extension \hat{f}_1 of \hat{f}_0 which we can use to parametrize V in the sense of Lemma 6.4.

In particular, there is a G -invariant sub bundle V_1 of $\Gamma^{(0,1)}(T_{\text{vert}}^* \mathbf{C}(\hat{f}_1) \otimes T_{\text{vert}} \hat{\mathbf{B}})$ so that given any family of curves \hat{h} in \mathcal{O} , there is a G -fold cover \hat{h}' of \hat{h} and a lift

of the G -equivariant map $\mathbf{C}(\hat{h}') \longrightarrow \mathbf{C}(\hat{f}_0)$ to a G -equivariant map

$$\begin{array}{ccc} \mathbf{C}(\hat{h}') & \xrightarrow{\psi_1} & \mathbf{C}(\hat{f}_1) \\ \downarrow & & \downarrow \\ \mathbf{F}(\hat{h}') & \longrightarrow & \mathbf{C}(\hat{f}_1) \end{array}$$

so that the pullback of V_1 using ψ_1 is $V(\hat{h}')$.

Consider V_1 as a vector bundle over $\mathbf{F}(\hat{f}_1)$. Let \hat{f}_2 be the family of curves defined using the following pullback diagram:

$$\begin{array}{ccccc} & & \hat{f}_2 & & \\ & \searrow & & \nearrow & \\ \mathbf{C}(\hat{f}_2) & \xrightarrow{r} & \mathbf{C}(\hat{f}_1) & \xrightarrow{\hat{f}_1} & \hat{\mathbf{B}} \\ \downarrow & & \downarrow & & \\ \mathbf{F}(\hat{f}_2) = V_1 & \longrightarrow & \mathbf{F}(\hat{f}_1) & & \end{array}$$

\hat{f}_1 may be regarded as the subfamily of \hat{f}_2 which corresponds to the zero section of the bundle $V_1 \longrightarrow \mathbf{F}(\hat{f}_1)$.

There is a G -equivariant tautological section θ of $\Gamma^{(0,1)} \left(T_{vert}^* \mathbf{C}(\hat{f}_2) \otimes T_{vert} \hat{\mathbf{B}} \right)$

$$\theta(f', v) := r^* v$$

This tautological section has the following property: given any family of curves \hat{h} in \mathcal{O} and a section $\theta_{\hat{h}}$ of $V(\hat{h})$, the canonical map $\psi_1 : \mathbf{C}(\hat{h}') \longrightarrow \mathbf{C}(\hat{f}_1)$ lifts to a canonical G -equivariant map

$$\begin{array}{ccc} \mathbf{C}(\hat{h}') & \xrightarrow{\psi_2} & \mathbf{C}(\hat{f}_2) \\ \downarrow & & \downarrow \\ \mathbf{F}(\hat{h}') & \longrightarrow & \mathbf{C}(\hat{f}_2) \end{array}$$

so that the pullback of θ to $\mathcal{Y}(\hat{h}')$ is equal to $\theta_{\hat{h}}$. A particular case of interest is when $\theta_{\hat{h}} = \bar{\partial}\hat{h}$.

Recall that \hat{f}_1 is defined by pulling back \hat{f}_0 and \hat{f}_2 is defined by pulling back \hat{f}_1 . Therefore the sections $\{s_i\}$ of $\mathbf{C}(\hat{f}_0) \longrightarrow \mathbf{F}(\hat{f}_0)$ pull back to sections $\{s'_i\}$ of $\mathbf{C}(\hat{f}_2) \longrightarrow \mathbf{F}(\hat{f}_2)$. As the set of sections $\{s_i\}$ is G -invariant and the maps $\hat{f}_2 \longrightarrow \hat{f}_1 \longrightarrow \hat{f}_0$ are G -equivariant, the set of sections $\{s'_i\}$ is G -invariant.

Choose a G -invariant trivialization (F, Φ) to associate to \hat{f}_2 , and use this trivialization and θ to define a simple perturbation of $\bar{\partial}$

$$\bar{\partial}' : X^{\infty,1}(\hat{f}_2) \longrightarrow \mathcal{Y}(\hat{f}_2)$$

$$\bar{\partial}'(\nu) := \bar{\partial}\nu - \Phi((\text{id}, F(\nu))^* \theta)$$

as in Example 3.5. Note that $\bar{\partial}'$ is a G -equivariant map.

Recall from Lemma 6.4 that we can choose $(\hat{f}_0/G, \{s_i\})$ so that $D\bar{\partial}$ is injective when restricted to sections of $f^* T_{vert} \hat{\mathbf{B}}$ which vanish at the image of $\{s_i\}$. As θ vanishes on $\mathbf{C}(f) \subset \mathbf{C}(\hat{f}_2)$, $D\bar{\partial}'$ is also injective when restricted to sections of $f^* T_{vert} \hat{\mathbf{B}}$ which vanish at the image of $\{s_i\}$. The space $X^{\infty,1}(f)$ is the space of these sections which vanish on the image of $\{s_i\}$. We may therefore choose a G -invariant obstruction bundle V_2 over \hat{f}_2 so that

$$D\bar{\partial}' : X^{\infty,1}(f) \longrightarrow \mathcal{Y}(f)$$

has image complementary to V_2 . Note that V_2 need not have anything to do with V .

Apply Theorem 3.6 to (\hat{f}_2, V_2) to obtain a unique section ν in $X^{\infty, \perp}(\hat{f}_2)$ defined in a neighborhood of f in \hat{f}_2 so that $\bar{\partial}'\nu$ is a section of V_2 . The uniqueness part of Theorem 3.6 and the fact that $\bar{\partial}'$ is G -equivariant imply that ν is G -equivariant.

Claim 6.7. $\bar{\partial}'\nu$ is transverse to the 0-section of V_2 at the curve f in \hat{f}_2 .

To prove Claim 6.7 above, we shall need to use the condition that

$$D\bar{\partial} : T_f \mathcal{M}^{st}(\hat{\mathbf{B}}) \longrightarrow \mathcal{Y}(f)$$

is transverse to V . This transversality and Lemma 2.12 imply that given any element v_0 of $V_2(f)$, there exists a family \hat{h} parametrized by \mathbb{R} containing f at 0, and a section $\theta_{\hat{h}}$ of $V(\hat{h})$ so that the derivative at 0 of $\bar{\partial}\hat{h} - \theta_{\hat{h}}$ is v_0 . Recall that the tautological section θ of $\Gamma^{(0,1)}(T_{vert}^* \mathbf{C}(\hat{f}_2) \otimes T_{vert} \hat{\mathbf{B}})$ is defined so that there exists a canonical lift of the G -equivariant map $\psi_1 : \mathbf{C}(\hat{h}') \longrightarrow \mathbf{C}$ to a map

$$\psi_2 : \mathbf{C}(\hat{h}') \longrightarrow \mathbf{C}(\hat{f}_2)$$

so that

$$\psi_2^* \theta = \theta_{\hat{h}}$$

(or more accurately, $\psi_2^* \theta$ is equal to the lift of $\theta_{\hat{h}}$ to $\mathbf{C}(\hat{h}')$). We may then express \hat{h}' using the trivialization (F, Φ) associated to \hat{f}_2 , so there is a section $\nu_{\hat{h}'}$ of $\psi_2^* \hat{f}_2^* T_{vert} \hat{\mathbf{B}}$ so that $\hat{h}' = F(\psi_2, \nu_{\hat{h}'})$. The map ψ_2 is constructed so that

$$\bar{\partial}'\nu_{\hat{h}'} = \Phi(\bar{\partial}\hat{h} - \theta_{\hat{h}})$$

In particular, $\bar{\partial}'\nu_{\hat{h}'}$ is zero at 0 and has first derivative equal to v_0 . Theorem 3.6 implies that there is a unique section $\nu'_{\hat{h}'}$ of $\psi_2^* \hat{f}_2^* T_{vert} \hat{\mathbf{B}}$ so that $\bar{\partial}'\nu'_{\hat{h}'} \in V_2$, and Corollary 3.7 implies that $\nu'_{\hat{h}'}$ and $\nu_{\hat{h}'}$ are equal to first order at 0. It follows that the derivative at 0 of $\bar{\partial}'\nu'_{\hat{h}'}$ is equal to v_0 . The uniqueness part of Theorem 3.6 implies that $\nu'_{\hat{h}'} = \psi_2^* \nu$, so v_0 must be in the image of the derivative of $\bar{\partial}'\nu$ at the curve f in \hat{f}_2 . As this argument holds for any v_0 in $V_2(f)$, it follows that $\bar{\partial}'\nu$ is transverse to 0 at $f = 0$, and the proof of Claim 6.7 is complete.

Note that given any family of curves \hat{h} in \mathcal{O} so that $\bar{\partial}\hat{h}$ is a section of $V(\hat{h})$, we have a canonical G -fold cover \hat{h}' of \hat{h} and map $\psi_2 : \mathbf{C}(\hat{h}') \longrightarrow \mathbf{C}(\hat{f}_2)$ so that $\psi_2^* \theta = \bar{\partial}\hat{h}'$. We may choose \mathcal{O} small enough so that everywhere in the image of ψ_2 restricted to families of curves in \mathcal{O} with $\bar{\partial}$ contained in V ,

- ν is defined,
- $\bar{\partial}'\nu$ is transverse to 0,
- and $D\bar{\partial}'$ is transverse to V_2 at ν .

Then the uniqueness property of Theorem 3.6 implies that on an open subset of \hat{h}' , $\hat{h}' = F(\nu) \circ \psi_2$. The open subset of \hat{h}' satisfying this last property includes any curve in \hat{h}' isomorphic to f . Lemma 2.8 implies that if we removed all curves not satisfying this property from \mathcal{O} , we would be left with an open substack, therefore, we may reduce the size of \mathcal{O} so that if h is any family of curves in \mathcal{O} with $\bar{\partial}h$ in V , then

$$\hat{h}' = F(\nu) \circ \psi_2$$

As $\psi_2^* \theta = \bar{\partial}\hat{h}'$, and $\bar{\partial}' = \bar{\partial} - \theta$, ψ_2 must have image contained in the subset of $\mathbf{C}(\hat{f}_2)$ where $\bar{\partial}'\nu = 0$.

Let \hat{f}_3 be the subfamily of $F(\nu)$ given by the intersection of $\bar{\partial}'\nu$ with 0 and restricted \mathcal{O} . The uniqueness property of ν implies that \hat{f}_3 is a G -invariant family of curves. So far, we have that given any family \hat{h} in \mathcal{O} so that $\bar{\partial}\hat{h}$ is a section of $V(h)$, there exists a G -fold cover \hat{h}' of \hat{h} and a unique map $\hat{h}' \rightarrow \hat{f}_3$ with the property that ψ_1 factorizes as

$$(9) \quad \begin{array}{ccccccc} \mathbf{C}(\hat{h}') & \longrightarrow & \mathbf{C}(\hat{f}_3) & \hookrightarrow & \mathbf{C}(\hat{f}_2) & \xrightarrow{r} & \mathbf{C}(\hat{f}_1) \\ & & & & \searrow & \nearrow & \\ & & & & \psi_1 & & \end{array}$$

This uniqueness implies that given any map $\hat{g} \rightarrow \hat{h}$, the map $\hat{g}' \rightarrow \hat{f}_3$ factorizes as $\hat{g}' \rightarrow \hat{h}' \rightarrow \hat{f}_3$, so we have a map of the substack $\{\bar{\partial} \subset V\} \subset \mathcal{O}$ to \hat{f}_3/G .

The map $\mathcal{O} \rightarrow \mathbf{C}(\hat{f}_1)/G$ applied to \hat{f}_3 gives a G -fold cover \hat{f}_3' of \hat{f}_3 and a map

$$\psi_1 : \mathbf{C}(\hat{f}_3') \rightarrow \mathbf{C}(\hat{f}_1)$$

For \mathcal{O} small enough, we may assume that \hat{f}_3' consists of $|G|$ copies of \hat{f}_3 , so ψ_1 above consists of $|G|$ maps $\mathbf{C}(\hat{f}_3') \rightarrow \mathbf{C}(\hat{f}_1)$. Both \hat{f}_3 and \hat{f}_1 have a canonical inclusion of the curve f . The fact from Lemma 6.4 that the core family map to $\mathbf{C}(\hat{f}_0)$ factorizes through the above map implies that exactly one of these maps $\mathbf{C}(\hat{f}_3') \rightarrow \mathbf{C}(\hat{f}_1)$ must be the identity on $\mathbf{C}(f)$. There is therefore a canonical lift

$$l : \hat{f}_3 \rightarrow \hat{f}_3'$$

and a canonical map

$$\psi_1' := \psi_1 \circ l : \mathbf{C}(\hat{f}_3) \rightarrow \mathbf{C}(\hat{f}_1)$$

so that ψ_1' is the identity on $\mathbf{C}(f)$.

Arguing as in the proof of Claim 6.5 gives that there is an action of $G \times G$ on \hat{f}_3' so that ψ_1 is invariant under the first factor and equivariant under the second factor, and $\hat{f}_3' \rightarrow \hat{f}_3$ is equivariant under the first factor and invariant under the second factor. The lift $l : \hat{f}_3 \rightarrow \hat{f}_3'$ is G -equivariant when the diagonal action of G is used on \hat{f}_3' , so the map ψ_1' is G -equivariant.

The map ψ_1' has the following important property: Given any family of curves \hat{h} in \mathcal{O} with $\bar{\partial}h$ a section of $V(h)$, the G -fold cover \hat{h}' of \hat{h} which maps to \hat{f}_3 is the same as the G -fold cover \hat{h}' of \hat{h} with a map $\psi_1 : \mathbf{C}(\hat{h}') \rightarrow \mathbf{C}(\hat{f}_1)$, and this map factorizes as

$$(10) \quad \begin{array}{ccccc} \mathbf{C}(\hat{h}') & \longrightarrow & \mathbf{C}(\hat{f}_3) & \xrightarrow{\psi_1'} & \mathbf{C}(\hat{f}_1) \\ & & \searrow & \nearrow & \\ & & \psi_1 & & \end{array}$$

The two factorizations of $\psi_1 : \mathbf{C}(\hat{h}') \rightarrow \mathbf{C}(\hat{f}_1)$ from the diagrams (9) and (10) imply that the map $\hat{h}' \rightarrow \hat{f}_3$ has image in the subset of $\mathbf{C}(\hat{f}_3)$ where r and ψ_1' coincide. Recall from Lemma 6.4 that ψ_1 followed by the projection $\mathbf{C}(\hat{f}_1) \rightarrow \mathbf{C}(\hat{f}_0)$ is the core family map to \hat{f}_0 . The map r followed by projection to $\mathbf{C}(\hat{f}_0)$ and quotiented by G is also the core family map from $\mathbf{C}(\hat{f}_2) \rightarrow \mathbf{C}(\hat{f}_0)/G$. This core family map is the same for \hat{f}_2 and $F(\nu)$, so the restriction of r to $\mathbf{C}(\hat{f}_3)$ followed by a quotient by G is also the core family map to $\mathbf{C}(\hat{f}_0)/G$. As both r and ψ_1' are the identity on the canonical inclusion of f in \hat{f}_3 , the step of taking a quotient by G is not necessary, and the following diagram commutes:

$$\begin{array}{ccc}
\mathbf{C}(\hat{f}_3) & \xrightarrow{\psi'_1} & \mathbf{C}(\hat{f}_1) \\
\downarrow r & & \downarrow \\
\mathbf{C}(\hat{f}_1) & \longrightarrow & \mathbf{C}(\hat{f}_0)
\end{array}$$

Claim 6.8. *At the curve f in \hat{f}_3 , the map*

$$(r, \psi'_1) : \mathbf{C}(\hat{f}_3) \longrightarrow \mathbf{C}(\hat{f}_1) \times_{\mathbf{C}(\hat{f}_0)} \mathbf{C}(\hat{f}_1)$$

is transverse to the diagonal section of $\mathbf{C}(\hat{f}_1) \times_{\mathbf{C}(\hat{f}_0)} \mathbf{C}(\hat{f}_1) \longrightarrow \mathbf{C}(\hat{f}_0)$.

To prove Claim 6.8, recall that there is an inclusion of \hat{f}_1 into \hat{f}_2 on which the tautological section θ is 0. The inverse image \hat{g} of f under the map $\hat{f}_1 \longrightarrow \hat{f}_0$ is holomorphic. Therefore, ν is 0 on the image of \hat{g} in \hat{f}_2 , so there is an inclusion $\hat{g} \longrightarrow \hat{f}_3 \subset \hat{f}_2$ lifting the inclusion $\hat{g} \longrightarrow \hat{f}_1$. As every curve in \hat{g} is isomorphic to f , ψ'_1 restricted to $\mathbf{C}(\hat{g})$ is constant. On the other hand, r restricted to $\mathbf{C}(\hat{g})$ is an isomorphism onto the fiber of $\mathbf{C}(\hat{f}_1) \longrightarrow \mathbf{C}(\hat{f}_0)$ over $\mathbf{C}(f)$, therefore (r, ψ') is transverse to the diagonal at f as required.

By restricting \hat{f}_3 to a possibly smaller neighborhood \mathcal{O} of f , Claim 6.8 implies that we may assume that the intersection of (r, ψ'_1) is transverse to the diagonal. Let \hat{f} be the restriction of \hat{f}_3 to the inverse image of the diagonal under (r, ψ'_1) . As the diagonal and (r, ψ'_1) are G -equivariant, \hat{f} is a G -invariant family.

As noted above, the map from $\{\bar{\partial} \subset V\} \subset \mathcal{O}$ to \hat{f}_3/G has image in the subset where r and ψ'_1 coincide, so we have a map ψ from $\{\bar{\partial} \subset V\} \subset \mathcal{O}$ to \hat{f}/G . To prove that \hat{f}/G represents this moduli stack of curves with $\bar{\partial}$ in V , we must verify the following:

Claim 6.9. *The map ψ applied to \hat{f} is the quotient map $\hat{f} \longrightarrow \hat{f}/G$.*

As noted above in equation (9), this map ψ on the moduli stack is constructed so that it is given as a unique lift of the map to $\mathbf{C}(\hat{f}_1)/G$. Applied to \hat{f} , we get a G -fold cover $\pi : \hat{f}' \longrightarrow \hat{f}$ with an equivariant pullback diagram

$$\begin{array}{ccc}
\hat{f}' & \xrightarrow{\psi} & \hat{f} \\
\downarrow \pi & & \downarrow \\
\hat{f} & \longrightarrow & \hat{f}/G
\end{array}$$

To prove claim 6.9, we must verify that the bottom arrow in the above diagram is the quotient map $\hat{f} \longrightarrow \hat{f}/G$. We shall achieve this by constructing an equivariant lift $l' : \hat{f} \longrightarrow \hat{f}'$ so that both $\pi \circ l'$ and $\psi \circ l'$ are the identity, implying that the bottom and righthand arrows in the above diagram are equal.

The defining property of ψ is that it is the unique map $\psi : \hat{f}' \longrightarrow \hat{f}$ with the property that $\psi_1 : \mathbf{C}(\hat{f}') \longrightarrow \mathbf{C}(\hat{f}_1)$ factorizes as

$$\begin{array}{ccccc}
\mathbf{C}(\hat{f}') & \xrightarrow{\psi} & \mathbf{C}(\hat{f}) & \xrightarrow{r} & \mathbf{C}(\hat{f}_1) \\
& & \searrow \psi_1 & \nearrow & \\
& & & &
\end{array}$$

On the other hand, as r coincides with ψ'_1 on \hat{f} , ψ_1 factorizes as

$$\begin{array}{ccccc} \mathbf{C}(\hat{f}') & \xrightarrow{\psi} & \mathbf{C}(\hat{f}) & \xrightarrow{\psi'_1} & \mathbf{C}(\hat{f}_1) \\ & & \searrow \psi_1 & \nearrow & \\ & & & & \end{array}$$

The fact that ψ_1 comes from a map of stacks and \hat{f} is a sub family of \hat{f}_3 implies that following equivariant diagram commutes

$$\begin{array}{ccccc} & & \psi_1 & & \\ & \searrow & & \searrow & \\ \mathbf{C}(\hat{f}') & \longrightarrow & \mathbf{C}(\hat{f}_3) & \xrightarrow{\psi_1} & \mathbf{C}(\hat{f}_1) \\ \downarrow \pi & & \downarrow & & \\ \mathbf{C}(\hat{f}) & \hookrightarrow & \mathbf{C}(\hat{f}_3) & & \end{array}$$

As with \hat{f}'_3 , there is a $G \times G$ action on \hat{f}' so that $\pi : \hat{f}' \rightarrow \hat{f}$ is equivariant with respect to the first factor, and invariant with respect to the second factor, and ψ is invariant with respect to the first factor and equivariant with respect to the second factor.

The canonical lift $l : \hat{f}_3 \rightarrow \hat{f}'_3$ which pulls back ψ_1 to ψ'_1 coincides with a canonical lift $l' : \hat{f} \rightarrow \hat{f}'$ so that the following diagram commutes

$$\begin{array}{ccccc} & & \psi_1 & & \\ & \searrow & & \searrow & \\ \mathbf{C}(\hat{f}') & \longrightarrow & \mathbf{C}(\hat{f}'_3) & \xrightarrow{\psi_1} & \mathbf{C}(\hat{f}_1) \\ \uparrow l' & & \uparrow l & \nearrow \psi' & \\ \mathbf{C}(\hat{f}) & \hookrightarrow & \mathbf{C}(\hat{f}_3) & & \end{array}$$

In particular, this implies that on the image of l' , π factorizes ψ_1

$$\begin{array}{ccccccc} \mathbf{C}(\hat{f}) & \xrightarrow{l'} & \mathbf{C}(\hat{f}') & \xrightarrow{\pi} & \mathbf{C}(\hat{f}) & \xrightarrow{r = \psi'_1} & \mathbf{C}(\hat{f}_1) \\ & & & & \searrow \psi_1 \circ l' & \nearrow & \end{array}$$

Therefore, on the image of l' , π must coincide with ψ , as this factorization is the defining property of $\psi : \hat{f}' \rightarrow \hat{f}$. Of course, $\pi \circ l'$ is the identity, so $\psi \circ l' : \mathbf{C}(\hat{f}) \rightarrow \mathbf{C}(\hat{f})$ is the identity too. Note that ψ is determined by this and the condition that it is invariant with respect to the first G action and equivariant with respect to the second G action. It follows that ψ applied to \hat{f} must be the quotient map $\hat{f} \rightarrow \hat{f}/G$. This completes the proof of Claim 6.9 and proof that \hat{f}/G represents the moduli stack $\{\bar{\partial} \subset V\} \subset \mathcal{O}$.

To complete the proof of our theorem, we still need to verify that the tangent space of $\mathbf{F}(\hat{f})$ at f is equal to $D\bar{\partial}^{-1}(V(f))$.

The following claim allows us to think of $\mathbf{C}(\hat{f})$ as embedded in both $\mathbf{C}(\hat{f}_1)$ and $\mathbf{C}(\hat{f}_0)$. It implies that any G -invariant section of $V(\hat{f})$ may be extended to a global section of V defined on a neighborhood of \hat{f} in \mathcal{M}^{st} , and similarly any G -invariant, $C^{\infty, \perp}$ function of $\mathbf{F}(\hat{f})$ may be extended to a G -invariant, $C^{\infty, \perp}$ function defined on a neighborhood of \hat{f} in \mathcal{M}^{st} .

Claim 6.10. *The maps*

$$\psi'_1 : \mathbf{C}(\hat{f}) \rightarrow \mathbf{C}(\hat{f}_1)$$

and

$$\mathbf{C}(\hat{f}) \xrightarrow{\psi'_1} \mathbf{C}(\hat{f}_1) \longrightarrow \mathbf{C}(\hat{f}_0)$$

are embeddings in a neighborhood of f .

To prove Claim 6.10, note that $\mathbf{F}(\hat{f}_3) \longrightarrow \mathbf{F}(\hat{f}_2)$ is an embedding, locally defined by transverse vanishing of some $C^{\infty,1}$ functions. (This was proved in Claim 6.7.) $\mathbf{F}(\hat{f}) \subset \mathbf{F}(\hat{f}_3)$ is defined as the subset where r and ψ' agree. These maps are proved to be transverse in Claim 6.8. As required by Lemma 6.4 part 2, $\mathbf{C}(\hat{f}_1) \longrightarrow \mathbf{C}(\hat{f}_0)$ is isomorphic to a vector bundle. The maps r and ψ' agree after composition with the map $\mathbf{C}(\hat{f}_1) \longrightarrow \mathbf{C}(\hat{f}_0)$, therefore $\mathbf{F}(\hat{f}) \subset \mathbf{F}(\hat{f}_2)$ is also an embedding locally defined by the transverse vanishing of some $C^{\infty,1}$ functions.

As $\mathbf{F}(\hat{f}_2)$ is isomorphic to a vector bundle over $\mathbf{F}(\hat{f}_1)$ which is isomorphic to a vector bundle over $\mathbf{F}(\hat{f}_0)$, to finish Claim 6.10 it now suffices to prove that the derivative of the map $\mathbf{F}(\hat{f}) \longrightarrow \mathbf{F}(\hat{f}_0)$ is injective at f . Let v be a vector in $T_f \mathbf{F}(\hat{f})$ which is sent to 0 in $T_f \mathbf{F}(\hat{f}_0)$. As this map $\mathbf{F}(\hat{f}) \longrightarrow \mathbf{F}(\hat{f}_0)$ corresponds to the map from \hat{f}_0 being a core family, it follows that v must be equal to a section of $f^* T_{\text{vert}} \hat{\mathbf{B}}$ vanishing on the extra marked points in the definition of the core family \hat{f}_0 . Lemma 6.4 part 3 specifies that $D\bar{\partial}^{-1}V(f)$ does not contain any nonzero such vector, but $D\bar{\partial}(v)$ must be in $V(f)$ because $\bar{\partial}\hat{f}$ is a section of $V(\hat{f})$. It follows that the image of v in $T_f \mathcal{M}^{st}$ is 0. In particular, $D\bar{\partial}(v) = 0$, therefore v is tangent to $\mathbf{F}(\hat{f}_1) \subset \mathbf{F}(\hat{f}_2)$. The projection of $T_f \mathbf{F}(\hat{f}_2)$ onto $T_f \mathbf{F}(\hat{f}_1)$ comes from the map r , which coincides on $T_f \mathbf{F}(\hat{f}) \subset T_f \mathbf{F}(\hat{f}_2)$ with the map coming from ψ'_1 . As ψ'_1 comes from a fiberwise holomorphic map of a neighborhood of f in \mathcal{M}^{st} , Lemma 2.13 implies that map $T_f \mathbf{F}(\hat{f}) \longrightarrow T_f \mathbf{F}(\hat{f}_1)$ factors through $T_f \mathcal{M}^{st}$, therefore the image of v in $T_f \mathbf{F}(\hat{f}_1)$ must be zero. As we have already established that v is tangent to $T_f \mathbf{F}(\hat{f}_1)$, it follows that v is the zero vector in $T_f \mathbf{F}(\hat{f})$ and the map $T_f \mathbf{F}(\hat{f}) \longrightarrow T_f \mathbf{F}(\hat{f}_0)$ is injective.

It follows that on some neighborhood of f in $\mathbf{F}(\hat{f})$, the map $\mathbf{F}(\hat{f}) \longrightarrow \mathbf{F}(\hat{f}_0)$ is an embedding, locally equal to the transverse vanishing of some $C^{\infty,1}$ functions. As $\mathbf{F}(\hat{f}_1) \longrightarrow \mathbf{F}(\hat{f}_0)$ is isomorphic to a vector bundle, the same holds for the map $\mathbf{F}(\hat{f}) \longrightarrow \mathbf{F}(\hat{f}_1)$. This completes the proof of Claim 6.10

To complete the proof of Theorem 6.6, it remains to prove the following

Claim 6.11. *The map*

$$T_f \mathbf{F}(\hat{f}) \longrightarrow T_f \mathcal{M}^{st}(\hat{\mathbf{B}})$$

corresponding to the derivative of \hat{f} at f is injective, and has image equal to

$$D\bar{\partial}^{-1}(V(f)) \subset T_f \mathcal{M}^{st}(\hat{\mathbf{B}})$$

To prove Claim 6.11, note that Claim 6.10 gives that the map $T_f \mathbf{F}(\hat{f}) \longrightarrow T_f \mathbf{F}(\hat{f}_0)$ is injective. As this map comes from the map to the core family \hat{f}_0 , Lemma 2.13 implies that it factors through $T_f \mathcal{M}^{st}$, therefore $T_f \mathbf{F}(\hat{f})$ injects into $T_f \mathcal{M}^{st}$.

So far, we have seen that $T_f \mathbf{F}(\hat{f}) \longrightarrow T_f \mathcal{M}^{st}$ is injective. Obviously, $T_f \mathbf{F}(\hat{f})$ has image contained inside $D\bar{\partial}^{-1}(V(f))$, so it remains to show that the image of $T_f \mathbf{F}(\hat{f})$ contains $D\bar{\partial}^{-1}(V(f))$. Given any vector v in $D\bar{\partial}^{-1}(V(f))$, Lemma 2.12 implies that there exists a family \hat{h} of curves in \mathcal{O} parametrized by \mathbb{R} so that f is the curve over 0 and the derivative of \hat{h} at 0 is equal to v . There exists a section $\theta_{\hat{h}}$ of $V(\hat{h})$ so that $\bar{\partial}\hat{h} - \theta_{\hat{h}}$ is tangent to the zero section at 0. Then there exists a map $\psi'_2 : \mathbf{C}(\hat{h}) \longrightarrow \mathbf{C}(\hat{f}_2)$ and a section $\nu_{\hat{h}}$ of $(\hat{f}_2 \circ \psi'_2)^* T_{\text{vert}} \hat{\mathbf{B}}$

which vanishes on the inverse image of the marked point sections $\{s_i\}$ so that $\hat{h} = F(\nu_{\hat{h}})$ and $\bar{\partial}'\nu_{\hat{h}}$ is tangent to the zero section at 0. Theorem 3.6 then implies that close to 0, there exists a section $\nu'_{\hat{h}}$ for which $\bar{\partial}'\nu'_{\hat{h}} = 0$, and Corollary 3.7 implies that $\nu'_{\hat{h}}$ is tangent to $\nu_{\hat{h}}$ at 0. The uniqueness part of Theorem 3.6 implies that $\nu'_{\hat{h}} = (\psi'_2)^*\nu$, so \hat{h} is tangent at 0 to the family \hat{f}_3 at f . This proves that the image of $T_f\mathbf{F}(\hat{f}_3) \rightarrow T_f\mathcal{M}^{st}$ contains $D\bar{\partial}^{-1}(V(f))$, so the image of $T_f\mathbf{F}(\hat{f})$ also contains $D\bar{\partial}^{-1}(V(f))$, and $T_f\mathbf{F}(\hat{f})$ is equal to $D\bar{\partial}^{-1}(V(f))$.

This completes the proof of Claim 6.11 and Theorem 6.6. \square

7. CONSTRUCTION OF AN EMBEDDED KURANISHI STRUCTURE

Throughout this section, we shall be assuming that the map $\mathcal{M} \rightarrow \mathbf{B}_0$ is proper when restricted to any connected component of \mathcal{M}^{st} . This compactness property for the moduli stack of holomorphic curves is proved for many targets \mathbf{B} in [11]. We need this assumption in order to use Lemma 5.10, and in order to construct our Kuranishi charts to give a locally finite cover of \mathcal{M} .

Lemma 7.1. *Let \mathcal{O} be an open neighborhood of a holomorphic curve $f \in \mathcal{M}^{st}$ with a $C^{\infty,1}$ submersion*

$$\Phi : \mathcal{O} \rightarrow \mathbf{X}$$

to an exploded manifold or exploded orbifold \mathbf{X} . Then there exists an open neighborhood \mathcal{U} of $f \in \mathcal{M}^{st}$ and on \mathcal{U} a simply generated complex subsheaf V of \mathcal{Y} so that

- (1) *V is strongly transverse to $\bar{\partial}$ at all holomorphic curves in \mathcal{U} ,*
- (2) *At any holomorphic curve f' in \mathcal{U} , $D\bar{\partial}$ restricted to the kernel of $T_{f'}\Phi$ is strongly transverse to $V(f')$, so for any $t \in [0, 1]$*

$$T_{f'}\Phi(((1-t)D\bar{\partial} + tD\bar{\partial}^{\mathbb{C}})^{-1}(V(f'))) = T_{\Phi(f')}\mathbf{X}$$

Suppose further that there are a finite number open subsets \mathcal{U}_i of \mathcal{M}^{st} on which are defined simply generated subsheaves V_i of \mathcal{Y} , and for each \mathcal{U}_i , there is a chosen substack C_i of $\mathcal{M} \cap \mathcal{U}_i$ for each i , so that C_i is a closed substack of \mathcal{M}^{st} . Then V may be modified so that in addition to the above conditions, for any holomorphic curve f' in $C_i \cap \mathcal{U}$, the intersection of $V_i(f')$ and $V(f')$ is 0.

Proof:

If f has domain \mathbf{T} , this lemma follows from Lemma 7.5 and the observations that precede it on page 72. We shall therefore assume that the domain of f is not \mathbf{T} .

Proposition 5.8 gives that there exists an open neighborhood \mathcal{U} of f in \mathcal{M}^{st} and a core family \hat{f}/G for \mathcal{U} containing f . Theorem 3.8 from page 25 implies that there exists a finite dimensional subspace of $\mathcal{Y}(f)$ which is strongly transverse to $\bar{\partial}$ in the sense of Definition 2.22 on page 18. As the codimension of $\ker T_f\Phi \subset T_f\mathcal{M}^{st}$ is finite, we may also construct our finite dimensional subspace of $\mathcal{Y}(f)$ to be strongly transverse to $D\bar{\partial}$ restricted to $\ker T_f\Phi$. We may assume that G preserves $[f]$ within $[\hat{f}]$. As $\mathcal{Y}(f)$ only depends on the smooth part of f , the action of G on $[f]$ gives an action of G on $\mathcal{Y}(f)$. As this action of G on $\mathcal{Y}(f)$ is complex, we may choose a finite dimensional, complex, G -invariant subspace $V(f)$ of $\mathcal{Y}(f)$ which is strongly transverse to $\bar{\partial}$ and transverse to $D\bar{\partial}(\ker T_f\Phi)$.

There exists a complex basis $\{v_1(f), \dots, v_n(f)\}$ for $V(f)$ so that the action of $g \in G$ in this basis is given by a $n \times n$ complex matrix A_g .

As the inclusion $\mathbf{C}(f) \rightarrow \mathbf{C}(\hat{f})$ is an isomorphism onto a fiber of $\mathbf{C}(\hat{f})$, there exist sections v'_i of $\Gamma^{(0,1)}(T_{vert}^* \mathbf{C}(\hat{f}) \otimes_{T_{vert}} \hat{\mathbf{B}})$ considered as a sheaf of $C^{\infty,1}(\mathbf{F}(\hat{f}), \mathbb{C})$ modules so that the pullback of v'_i to $\mathcal{Y}(f)$ is $v_i(f)$. (This pullback is interpreted as in Definition 2.19.) If v' indicates the vector with components v'_i , then define

$$v := \sum_{g \in G} A_g^{-1} g * v'$$

Note that

$$g * v = A_g v$$

so the the sheaf of $C^{\infty,1}(\mathbf{F}(\hat{f}), \mathbb{C})$ modules generated by the components of v is G -invariant. Restricted to $\mathbf{C}(f)$, $g * v' = A_g v'$, so the pullback of the i th component of v to $\mathcal{Y}(f)$ is equal to $|G| v_i(f)$. It follows that the complex subsheaf V of \mathcal{Y} which is generated by the pullback of this sheaf of $C^{\infty,1}(\mathbf{F}(\hat{f}), \mathbb{C})$ modules is equal to $V(f)$ at f , and is simply generated on some neighborhood \mathcal{U} of f in the sense of Definition 2.20.

We have chosen V so that $V(f)$ is strongly transverse to $D\bar{\partial}$ restricted to $(\ker T_f \Phi)$. In particular, as $T_f \Phi$ is surjective, this is equivalent to requiring that $V(f)$ is strongly transverse to $D\bar{\partial}$, and that

$$T_f \Phi(((1-t)D\bar{\partial} + tD\bar{\partial}^{\mathbb{C}})^{-1} V(f)) = T_{\Phi(f)} \mathbf{X}$$

Theorem 6.6 states that if \mathcal{U} is small enough the moduli stack of curves \hat{h} in \mathcal{U} so that $\bar{\partial}h$ is a section of $V(h)$ is represented by the quotient of some family \hat{f}' of curves by an automorphism group G' . Theorem 3.6 then implies that at all holomorphic curves in a neighborhood of f in \hat{f}' , V is strongly transverse to $D\bar{\partial}$. Lemma 8.6 on page 78 states that in a neighborhood of f in \hat{f}' there exists a family of sub vectorbundles K_t of $T_{\hat{f}'} \mathcal{M}^{st} \downarrow_{\hat{\mathbf{B}}}$ which restrict to holomorphic curves f' to be

$$K_t(f') = ((1-t)D\bar{\partial} + tD\bar{\partial}^{\mathbb{C}})^{-1} V(f')$$

We have that $T_f \Phi$ is surjective restricted to $K_t(f)$, therefore the same holds for a neighborhood of f . It follows that so long as we have chosen \hat{f}' and \mathcal{U} small enough, at any holomorphic curve f' in \mathcal{U} and therefore in \hat{f}' , $D\bar{\partial}$ restricted to $\ker T_{f'} \Phi$ is strongly transverse to V .

It remains to prove that V may be modified so that for any holomorphic curve h in C_i , $V(h) \cap V_i(h) = 0$. For applications of this lemma, it is important that the domain \mathcal{U} of definition of this modified V does not depend on \mathcal{U}_i and V_i . Choose our \mathcal{U} so that \mathcal{M} intersected with the closure, $\bar{\mathcal{U}}$, of \mathcal{U} , is compact, and so that that V is still defined and satisfies the required transversality conditions on a larger neighborhood \mathcal{U}' so that \mathcal{U}' is an open neighborhood of $\bar{\mathcal{U}} \cap \mathcal{M}$. (Lemma 5.10 implies that such a reduction of the size of \mathcal{U} is possible.) We shall use the convention that \hat{f}/G is the core family for this larger \mathcal{U}'

Recall that V is pulled back from a G -invariant complex subsheaf of $\Gamma^{(0,1)}(T_{vert}^* \mathbf{C}(\hat{f}) \otimes_{T_{vert}} \hat{\mathbf{B}})$. This subsheaf is considered as a sheaf of $C^{\infty,1}(\mathbf{F}(\hat{f}), \mathbb{C})$ modules, but there is also an action of $C^{\infty,1}(\mathbf{C}(\hat{f}) \times \hat{\mathbf{B}}, \mathbb{C})$ by multiplication on $\Gamma^{(0,1)}(T_{vert}^* \mathbf{C}(\hat{f}) \otimes_{T_{vert}} \hat{\mathbf{B}})$. Multiplication by any G -invariant, \mathbb{C}^* valued, function m on $\mathbf{C}(\hat{f}) \times \hat{\mathbf{B}}$ sends V to some other complex, simply generated subsheaf mV of \mathcal{Y} .

Consider a family m_t of such G -invariant \mathbb{C}^* valued functions on $\mathbf{C}(\hat{f}) \times \hat{\mathbf{B}}$ parametrized by \mathbb{R} so that $m_0 = 1$. Then we may consider $(\hat{f} \times \mathbb{R}, G)$ to be a core family for $\mathcal{U}' \times \mathbb{R}$ in $\mathcal{M}^{st}(\hat{\mathbf{B}} \times \mathbb{R})$, and $m_t V$ is a simply generated complex subsheaf

of \mathcal{Y} on $\mathcal{U}' \times \mathbb{R}$. As proved above, if $m_t V$ satisfies the required transversality conditions at any holomorphic curve h , it satisfies these transversality conditions at all holomorphic curves in a neighborhood of h . As the required transversality conditions hold for all holomorphic curves in \mathcal{U}' , and $\mathcal{M} \cap \bar{\mathcal{U}}$ is compact and contained in \mathcal{U}' , it follows that for some neighborhood O of 0 in \mathbb{R} , the required transversality conditions hold for all holomorphic curves in $\mathcal{U} \times O$.

Let h be a holomorphic curve in $\bar{\mathcal{U}} \cap \mathcal{M}$. Given any nonzero $v \in V(h)$, as the map $\mathbf{C}(h) \rightarrow \mathbf{C}(\hat{f})/G$ is an immersion, there exists some G -invariant function m_v on $\mathbf{C}(\hat{f}) \times \hat{\mathbf{B}}$ so that if $h \in \mathcal{U}_i$, then

$$m_v v \notin V(h) \oplus V_i(h)$$

It follows that for all curves h' and $v' \in V(h')$ within a neighborhood of (h, v) in $V(\hat{f}')$

$$m_v v' \notin V(h') \oplus_i V_i(h')$$

The compactness of $C_i \cap \bar{\mathcal{U}}$ and the fact that V is finitely generated then imply that there exists some m so that for all holomorphic curves $h \in C_i$, and nonzero $v \in V(h)$,

$$mv \notin V(h) \oplus V_i(h)$$

Then for $t > 0$ small enough,

$$e^{tm} v \notin \oplus V_i(h)$$

so $e^{tm} V(h) \cap \oplus V_i(h) = 0$ for all $h \in \mathcal{M} \cap \mathcal{U}$. As our transversality conditions also hold for t small enough, it follows that $e^{tm} V$ is a modification of V with the required properties for any $t > 0$ small enough. \square

We are now ready to prove the existence of embedded Kuranishi structures defined in Definition 2.29 on page 21.

Theorem 7.2. *Suppose that the map $\mathcal{M} \rightarrow \mathbf{B}_0$ is proper when restricted to any connected component of \mathcal{M}^{st} . Then there exists an embedded Kuranishi structure on \mathcal{M} .*

Given any submersion $\Phi : \mathcal{M}^{st} \rightarrow \mathbf{X}$ where \mathbf{X} is an exploded manifold or orbifold, all Kuranishi charts can be chosen to be Φ -submersive in the sense of Definition 2.25.

This embedded Kuranishi structure may be chosen to include any countable, locally finite, compatible collection of extendible, Φ -submersive Kuranishi charts $\{(\mathcal{U}_i, V_i, \hat{f}_i/G_i)\}$ satisfying the above submersion condition.

Proof:

We are given a locally finite, countable collection $\{(\mathcal{U}_i, V_i, \hat{f}_i/G_i)\}$ of extendible, Φ -submersive Kuranishi charts. We shall assume that these charts are indexed by negative integers k , leaving positive integers free for constructing the rest of our Kuranishi charts. As specified in definitions 2.27 and 2.28, there are extensions

$$(\mathcal{U}_k^\#, V_k, \hat{f}_k^\#/G_k)$$

of $(\mathcal{U}_k, V_k, \hat{f}_k/G_k)$ so that each pair of these extended Kuranishi charts are compatible, and $\{\mathcal{U}_k^\#\}$ is a locally finite collection of substacks of \mathcal{M}^{st} . In particular, as specified by Definition 2.28, each holomorphic curve f has a neighborhood which intersects only finitely many $\mathcal{U}_i^\#$. Definition 2.27 gives that there is a continuous function $\rho_k : \mathcal{U}_k^\# \rightarrow (0, 1]$ so that $\mathcal{U}_k = \{\rho_k > 0.5\} \subset \mathcal{U}_k^\#$ and for any $t > 0$, any holomorphic curve in the closure in \mathcal{M}^{st} of $\{\rho_k > t\}$ is contained in $\{\rho_k \geq t\} \subset \mathcal{U}_k^\#$. We shall use the restriction of $(\mathcal{U}_k^\#, V_k, \hat{f}_k^\#/G_k)$ to $\{\rho_k > 0.4\}$ instead of our original extension.

For any holomorphic curve f , we have that there exists a neighborhood of f which intersects only finitely many \mathcal{U}_k^\sharp , and which does not intersect $\{\rho_k > 0.1\}$ if f is not in \mathcal{U}_k^\sharp .

Lemma 7.1 and Lemma 5.10 imply that each holomorphic curve f has a neighborhood \mathcal{O} with a continuous function $\rho : \mathcal{O} \rightarrow [0, 1]$ so that

- all holomorphic curves in the closure of $\{\rho > 0\}$ are contained in \mathcal{O} ,
- $\rho(f) = 1$,
- \mathcal{O} satisfies the conditions of Lemma 7.1,
- \mathcal{O} intersects only finitely many \mathcal{U}_k^\sharp , and intersects $\{\rho_k > 0.1\}$ only if f is in \mathcal{U}_k^\sharp .

Our assumption that $\mathcal{M} \rightarrow \mathbf{B}_0$ is proper restricted to each connected component of \mathcal{M}^{st} implies that these (\mathcal{O}, ρ) may be chosen so that there is a countable collection $\{(\mathcal{O}_i, \rho_i)\}$ of them indexed by natural numbers so that the sets $\{\rho_i > 0.5\} \subset \mathcal{O}_i$ cover \mathcal{M} , and each \mathcal{O}_i intersects only finitely many \mathcal{O}_j .

For each $i \in \mathbb{N}$ in turn, the second part of Lemma 7.1 implies that on \mathcal{O}_i , we may choose a simply generated complex subsheaf V_i of \mathcal{Y} satisfying the transversality conditions of Lemma 7.1 so that given any holomorphic curve f in \mathcal{O}_i , any $V_k(f)$ for which $\rho_k \geq 0.1$ and the $V_j(f)$ so that $0 < j \leq i$ and $\rho_j \geq 0.1$ are linearly independent in the sense that $V_k(f) \oplus_j V_j(f) \subset \mathcal{Y}(f)$ has dimension equal to the sum of the dimensions of the individual $V_j(f)$ and $V_k(f)$.

For any set A of negative integers, and nonempty $I \subset \mathbb{N}$, define the sheaf

$$V_{A,I} := \oplus_{k \in A} V_k \oplus_{i \in I} V_i$$

on $\cap_{k \in A} \mathcal{U}_k^\sharp \cap_{i \in I} \mathcal{O}_i$. Because the V_k are subsheaves of each other, the maximum dimension which $V_{A,I}$ can be is

$$\dim V_{A,I} = \max_{k \in A} \dim V_k + \sum_{i \in I} \dim V_i$$

Lemma 2.8 implies that the substack on which the dimension of $V_{A,I}$ is maximal is open. Let $\mathcal{O}_{A,I}$ denote this open substack. As noted above, $\mathcal{O}_{A,I}$ is an open neighborhood of the holomorphic curves for which $\rho_j \geq 0.1$ for $j \in A \cup I$.

As we want to define compatible Kuranishi charts, we must determine where to use $V_{A,I}$ carefully. In particular, we shall use $V_{A,I}$ on an open substack $\mathcal{O}'_{A \cup I} \subset \mathcal{O}_{A,I}$ with the following properties:

- (1) If $f \in \mathcal{O}'_S$ and $\rho_j(f) > 0.4$, then $j \in S$
- (2) If $f \in \mathcal{O}'_S$ and ρ_j is not defined at f or $\rho_j \leq 0.1$, then $j \notin S$.
- (3) \mathcal{O}'_S intersects $\mathcal{O}'_{S'}$ nontrivially only if $S \subset S'$ or $S' \subset S$

We need to define \mathcal{O}'_S satisfying the above for finite subsets $S \subset \mathbb{Z}$ which contain at least one natural number. Let n_S be the number of j' so that $\rho_{j'}(f) \geq 0.1$ for some f on which $\rho_j \geq 1$ for at least one $j \in S$. Note that we constructed \mathcal{O}_i so that n_S is finite. Now define $\mathcal{O}'_{A \cup I}$ to be the interior of the set

$$\mathcal{O}''_{A \cup I} := \left\{ n_{A \cup I} \left(\min \left(0.4, \min_{j \in I \cup A} \rho_j \right) - \max \left(0.1, \max_{j' \notin I \cup A} \rho_{j'} \right) \right) > 0.1 \right\} \subset \mathcal{O}_{A,I}$$

In the above, we set ρ_j to be 0 where it is not yet defined. As the ρ_j do not necessarily extend to be continuous functions on \mathcal{M}^{st} , the above inequality does not necessarily define an open substack, and we must take \mathcal{O}'_S to be its interior. Each of the above required properties of \mathcal{O}'_S follows immediately for \mathcal{O}''_S , and therefore they also hold for \mathcal{O}'_S .

Claim 7.3. $\{\mathcal{O}'_S\}$ is an open cover of $\mathcal{M} \subset \mathcal{M}^{st}$.

To prove claim 7.3, we must show that each stable holomorphic curve f is in \mathcal{O}'_S for some S . We already know that for some $i \in \mathbb{N}$, $\rho_i(f) > 0.5$. There are at most n_i j 's so that $\rho_j(f) \geq 0.1$, therefore, there exists some set S containing i so that

$$n_i \left(\min \left(0.4, \min_{j \in A} \rho_j(f) \right) - \max \left(0.1, \max_{j' \notin S} \rho_{j'}(f) \right) \right) \geq 0.3$$

As by definition, $n_S \geq n_i$ when $i \in S$, f is in \mathcal{O}''_S . It remains to verify that f is in the interior of \mathcal{O}''_S . If f was in the boundary of \mathcal{O}''_S , then there would be some j' so that $\rho_{j'}$ was not defined at f , but f was in the closure of $\{\rho_{j'} > 0.1\}$. One of our conditions on $\rho_{j'}$ is that this is not possible for holomorphic curves f , therefore no holomorphic curve f is in the boundary of \mathcal{O}''_S . Therefore $f \in \mathcal{O}'_S$ and Claim 7.3 is proven.

We may now construct our Kuranishi charts.

Note that restricted to $\mathcal{O}'_{A \cup I}$, $V_{A,I}$ is a simply generated complex subsheaf of \mathcal{Y} which is strongly transverse to $D\bar{\partial}$ at all holomorphic curves. Theorem 6.6 then implies that each holomorphic curve f in $\mathcal{O}'_{A \cup I}$, has an open neighborhood \mathcal{U} on which the stack of curves with $\bar{\partial}$ in $V_{A,I}$ is locally represented by some \hat{f}/G . For \mathcal{U} small enough, $(\mathcal{U}, V_{A,I}, \hat{f}/G)$ is then a Kuranishi chart containing f .

One of the transversality conditions from Lemma 7.1 is that at holomorphic curves f , $D\bar{\partial}$ restricted to $\ker T_f \Phi$ is strongly transverse to V_i , which implies that by choosing \mathcal{U} small enough if necessary, we may assume that \hat{f} is Φ -submersive in the sense of Definition 2.25.

Claim 6.10 implies that if \mathcal{U} is chosen small enough \hat{f}/G may be regarded as embedded in a core family for \mathcal{U} . It follows that there exists a continuous function $\rho : \mathcal{U} \rightarrow [0, 1]$ which is 1 at f and which has compact support on \hat{f}/G . Lemma 5.10 implies furthermore that ρ may be chosen so that any holomorphic curve in the closure of $\{\rho > 0\}$ is contained in \mathcal{U} . Restricting $(\mathcal{U}, V_{A,I}, \hat{f}/G)$ to $\{\rho > 0.5\}$ gives a Kuranishi chart with an extension to $\{\rho > 0\}$. Condition 3 on $\mathcal{O}'_{A \cup I}$ implies that all such charts are compatible, and condition 1 of $\mathcal{O}'_{A \cup I}$ implies that any such chart is compatible with any of our original $(\mathcal{U}_k, V_k, \hat{f}_k/G_k)$ on the extension where $\rho_k > 0.4$.

Our properness assumption on $\mathcal{M} \rightarrow \mathbf{B}_0$ implies that \mathcal{M} has an exhaustion by compact substacks, therefore we may choose a countable, locally finite collection of extendible Kuranishi charts of the above type which cover \mathcal{M} . This collection of Kuranishi charts together with our original collection of Kuranishi charts is our required embedded Kuranishi structure □

Corollary 7.4. Any two embedded Kuranishi structures on $\mathcal{M}(\hat{\mathbf{B}})$ are homotopic in the sense that there exists an embedded Kuranishi structure on $\mathcal{M}(\hat{\mathbf{B}} \times \mathbb{R})$ which pulls back to the two given embedded Kuranishi structures via the inclusions of $\hat{\mathbf{B}}$ over the points 0 and 1 in \mathbb{R} .

If there is a submersion

$$\Phi : \mathcal{M}^{st}(\hat{\mathbf{B}}) \rightarrow \mathbf{X}$$

and the two original embedded Kuranishi structures are Φ -submersive, then the homotopy may be chosen to be Φ' submersive where Φ' is given by the composition

$$\mathcal{M}^{st}(\hat{\mathbf{B}} \times \mathbb{R}) \xrightarrow{\quad \quad \quad \Phi' \quad \quad \quad} \mathcal{M}^{st}(\hat{\mathbf{B}}) \xrightarrow{\quad \Phi \quad} \mathbf{X}$$

Proof:

Pull back the first embedded Kuranishi structure to a collection of Kuranishi charts over $(-\infty, \frac{1}{3})$ and pull back the second embedded Kuranishi structure to a collection of Kuranishi charts over $(\frac{2}{3}, \infty)$. Together, these charts give a countable, locally finite, extendible collection of Φ' -submersive, compatible Kuranishi charts on $\mathcal{M}^{st}(\hat{\mathbf{B}} \times \mathbb{R})$.

Theorem 7.2 implies that we may expand this collection of Kuranishi charts to a Φ' -submersive embedded Kuranishi structure. \square

7.1. The case of curves with domain \mathbf{T} .

Observe the following:

- If any curve in a connected family of curves \hat{f} has domain \mathbf{T} , then all curves in \hat{f} have domain \mathbf{T} .
- If any curve in a connected component of \mathcal{M}^{st} has domain \mathbf{T} , then all curves in that connected component have domain \mathbf{T} .
- If a curve f has domain \mathbf{T} , then f is holomorphic and $\mathcal{Y}(f)$ is trivial.
- As the smooth part of \mathbf{T} is a single point, a curve f in \mathbf{B} with domain \mathbf{T} is always contained in a single coordinate chart, and any curve in a neighborhood of f within \mathcal{M}^{st} is also contained in the same coordinate chart.
- Any coordinate chart on $\hat{\mathbf{B}}$ containing a stable curve f with domain \mathbf{T} may be written in the form $U \times \mathbf{T}$ so that

$$f(\tilde{z}) = (u, ct^a \tilde{z}^n)$$

where $u \in U$ and $ct^a \in \mathbb{C}^* t^{\mathbb{R}}$ are constant, and n is a positive integer. All nearby curves f' will be in the same form, with different constants, but the same integer n .

Lemma 7.5. *If n is a positive integer, and U is a connected exploded manifold, the moduli stack of curves f in $U \times \mathbf{T}$ in the form*

$$f(\tilde{z}) = (u, ct^a \tilde{z}^n)$$

is represented by quotient of the family of curves

$$\begin{array}{ccc} U \times \mathbf{T} & \xrightarrow{(\text{id}, \tilde{z}^n)} & U \times \mathbf{T} \\ \downarrow & & \\ U & & \end{array}$$

by its automorphism group, \mathbb{Z}_n .

Proof: Call this family of curves \hat{f}_0 . The automorphism group of \hat{f}_0 is \mathbb{Z}_n , which acts trivially on $U = \mathbf{F}(\hat{f}_0)$, and acts by multiplying the \mathbf{T} fibers of $\mathbf{C}(\hat{f}_0)$ by n th roots of unity.

Given any family \hat{f} of curves of the required type in $U \times \mathbf{T}$, define $\mathbf{C}(\hat{f})$ to be the n -fold cover of $\mathbf{C}(\hat{f})$ given by the fiber product

$$\mathbf{C}(\hat{f}') := \mathbf{C}(\hat{f}) \times_{\hat{f} \times \hat{f}_0} U \times \mathbf{T}$$

and define $\mathbf{F}(\hat{f}')$ to be the fiber product of \hat{f} with the inclusion of U into $U \times \mathbf{T}$ which is the identity on U and maps to the constant $1t^0$ on the \mathbf{T} coordinate.

$$\mathbf{F}'(\hat{f}) := \mathbf{C}(\hat{f}) \times_{\hat{f} \times (\text{id}, 1t^0)} U$$

The map $U \times \mathbf{T} \rightarrow U$ induces a map $\mathbf{C}(\hat{f}') \rightarrow \mathbf{F}(\hat{f}')$ which makes $\mathbf{C}(\hat{f}') \rightarrow \mathbf{F}(\hat{f}')$ the family of \mathbf{T} 's pulled back from the diagram

$$\begin{array}{ccc} \mathbf{C}(\hat{f}') & \longrightarrow & \mathbf{C}(\hat{f}_0) \\ \downarrow & & \downarrow \\ \mathbf{F}(\hat{f}') & \longrightarrow & \mathbf{F}(\hat{f}_0) \end{array}$$

Pulling back \hat{f}_0 via the map $\mathbf{C}(\hat{f}') \rightarrow \mathbf{C}(\hat{f}_0)$ gives a family of curves \hat{f}' with maps

$$\begin{array}{ccc} \hat{f}' & \longrightarrow & \hat{f}_0 \\ \downarrow & & \\ \hat{f} & & \end{array}$$

There are two actions of \mathbb{Z}_n on \hat{f}' , induced by multiplying fibers of $\mathbf{C}(\hat{f})$ or $\mathbf{C}(\hat{f}_0)$ by n th roots of unity. The map $\hat{f}' \rightarrow \hat{f}_0$ is invariant under the first action and equivariant under the second action, whereas the map $\hat{f}' \rightarrow \hat{f}$ is equivariant under the first action and invariant under the second action. This second action makes $\hat{f}' \rightarrow \hat{f}$ a \mathbb{Z}_n -fold cover.

Now suppose that \hat{f}'' is some other \mathbb{Z}_n -fold cover of \hat{f} with a \mathbb{Z}_n -equivariant map $\hat{f}'' \rightarrow \hat{f}_0$. The construction of $\mathbf{C}(\hat{f}')$ as a fiber product gives a natural map

$$\hat{f}'' \rightarrow \hat{f}'$$

which must commute with the maps to \hat{f} and \hat{f}_0 , and be \mathbb{Z}_n equivariant because the action of \mathbb{Z}_n on $\mathbf{C}(\hat{f}_0)$ is free. It follows that $\hat{f}'' \rightarrow \hat{f}'$ is an equivariant isomorphism, so \hat{f}' is the unique \mathbb{Z}_n -fold cover of \hat{f} with a \mathbb{Z}_n equivariant map $\hat{f}' \rightarrow \hat{f}_0$.

Therefore our moduli stack of curves is represented by \hat{f}_0/\mathbb{Z}_n as required. \square

8. RELATIVE COMPLEX STRUCTURE AND ORIENTATION OF KURANISHI CHARTS

Suppose that $(\mathcal{U}, V, \hat{f}/G)$ is a Kuranishi chart. Recall that in the situation of a family of targets, $\hat{\mathbf{B}} \rightarrow \mathbf{B}_0$, the map

$$\mathbf{F}(\hat{f}) \rightarrow \mathbf{B}_0$$

is a submersion. In this section, we construct a canonical homotopy class of G -invariant complex structure on $T_f \mathbf{F}(\hat{f}) \downarrow_{\mathbf{B}_0}$ for all holomorphic curves f in \hat{f} . On a neighborhood of the holomorphic curves in \hat{f} , this defines a canonical orientation of $\mathbf{F}(\hat{f})$ relative to \mathbf{B}_0 .

Recall that if f is holomorphic, $T_f \mathcal{M}^{st} \downarrow_{\mathbf{B}_0}$ is complex, so it makes sense to talk of the complex linear part $D\bar{\partial}^{\mathbb{C}}$ of the linear map $D\bar{\partial} : T_f \mathcal{M}^{st} \downarrow_{\mathbf{B}_0} \rightarrow \mathcal{Y}(f)$. Suppose that $(\mathcal{U}_i, V_i, \hat{f}_i/G_i)$ is a Kuranishi chart containing the holomorphic curve f . Define

$$K_{i,t}(f) = ((1-t)D\bar{\partial} + tD\bar{\partial}^{\mathbb{C}})^{-1}(V_i(f)) \subset T_f \mathcal{M}^{st} \downarrow_{\mathbf{B}_0}$$

Theorem 3.8 and the condition that $\bar{\partial}$ is strongly transverse to V_i at f implies that $K_{i,t}(f)$ for $t \in [0, 1]$ is a smooth family of vector subspaces of $T_f\mathcal{M}^{st}$. Theorem 6.6 implies that $K_{i,0}(f)$ is equal to $T_f\mathbf{F}(\hat{f})\downarrow_{\mathbf{B}_0}$.

As $K_{i,1}(f)$ is the inverse image of the complex vector space $V_i(f)$ under a complex map, it is a complex subspace of $T_f\mathcal{M}^{st}\downarrow_{\mathbf{B}_0}$. Therefore, there is a canonical homotopy class of complex structure on $K_{i,0}(f)$ consisting of complex structures homotopic to the complex structure on $K_{i,1}(f)$. Of course, this gives a canonical orientation of $K_{i,0}(f)$, which in turn gives a canonical orientation of \hat{f}_i relative to \mathbf{B}_0 at f .

In this section, we shall show that $K_{i,t}$ may be regarded as a family of G_i -invariant, $C^{\infty,1}$ sub-vectorbundles of $T_{\hat{f}_i}\mathcal{M}^{st}\downarrow_{\mathbf{B}_0}$ restricted to $\{\bar{\partial}\hat{f}_i = 0\}$, with a canonical, G_i -invariant, $C^{\infty,1}$ complex structure on $K_{i,t}$. In Proposition 8.10, we shall show that $K_{i,t}$ may be identified for all t in a way which is G_i invariant, compatible with all inclusions $K_{i,t} \subset K_{j,t}$, constant on \mathbb{R} -nil vectors, and compatible with any chosen submersion $\Phi : \mathcal{M}^{st} \rightarrow \mathbf{X}$. This allows a kind of global construction of a canonical homotopy class of complex structure on $T_f\mathbf{F}(\hat{f}_i)\downarrow_{\mathbf{B}_0}$ at all holomorphic curves f .

The \mathbb{R} -nil vectors in $T_f\mathcal{M}^{st}\downarrow_{\mathbf{B}_0}$ (which are the vectors which act trivially as derivations on $C^{\infty,1}$ functions), form a complex linear subspace of $T_f\mathcal{M}^{st}\downarrow_{\mathbf{B}_0}$ which is in the kernel of $D\bar{\partial}$ and therefore always contained in $K_{i,t}(f)$.

Lemma 8.1. *Suppose that f is a holomorphic curve contained in \hat{f} and $T_f\mathbf{F}(\hat{f}) \rightarrow T_f\mathcal{M}^{st}$ is injective. Then the \mathbb{R} -nil vectors from $T_f\mathbf{F}(\hat{f})\downarrow_{\mathbf{B}_0}$ form a complex linear subspace of $T_f\mathcal{M}^{st}\downarrow_{\mathbf{B}_0}$ which is contained in the kernel of $D\bar{\partial} : T_f\mathcal{M}^{st}\downarrow_{\mathbf{B}_0} \rightarrow \mathcal{Y}(f)$.*

Proof:

Without losing generality, we may restrict to the case that $[\mathbf{F}(\hat{f})]$ is a single point. As in this case each curve in \hat{f} is holomorphic, it follows that $T_f\mathbf{F}(\hat{f})\downarrow_{\mathbf{B}_0}$ is in the kernel of $D\bar{\partial}$. As we are dealing with tangent spaces relative to \mathbf{B}_0 in this lemma, it suffices to consider the case when \mathbf{B}_0 is a point, so we may talk of $T\mathcal{M}^{st}(\mathbf{B})$ instead of $T\mathcal{M}^{st}(\hat{\mathbf{B}})\downarrow_{\mathbf{B}_0}$.

There exists a unique complex structure \mathbf{j} on $\mathbf{C}(\hat{f})$ which extends the given fiberwise almost complex structure. In particular, the exploded functions on $\mathbf{C}(\hat{f})$ which are holomorphic with respect to the fiberwise almost complex structure in local coordinates are always equal to some monomial times a holomorphic \mathbb{C}^* valued function on the smooth part of the coordinate chart. These exploded functions define a sheaf of holomorphic exploded functions on $\mathbf{C}(\hat{f})$ and give the canonical complex structure \mathbf{j} on $\mathbf{C}(\hat{f})$. As the smooth part of $\mathbf{F}(\hat{f})$ is a single point, there is a unique complex structure on $\mathbf{F}(\hat{f})$.

The map $\mathbf{C}(\hat{f}) \rightarrow \mathbf{F}(\hat{f})$ is holomorphic. In fact, any $C^{\infty,1}$ map from $\mathbf{C}(\hat{f})$ which is holomorphic with respect to the fiberwise almost complex structure must also be holomorphic with respect to this canonical complex structure \mathbf{j} because any $C^{\infty,1}$ map must be holomorphic restricted to \mathbb{R} -nil vectors, and the tangent space of $\mathbf{C}(\hat{f})$ is spanned by the vertical tangent space and \mathbb{R} -nil vectors. In particular, the map \hat{f} is \mathbf{j} -holomorphic.

We have established that \hat{f} is a holomorphic family of curves. We shall now check that the map $T_f\mathbf{F}(\hat{f}) \rightarrow T_f\mathcal{M}^{st}$ is complex linear. Let v be the lift of any vector field on $\mathbf{F}(\hat{f})$ to vector field on $\mathbf{C}(\hat{f})$. As \mathbf{j} is integrable, calculation in \mathbb{C}^n gives that

$$\mathbf{j} \circ L_v \mathbf{j} = L_{\mathbf{j}v} \mathbf{j}$$

Therefore,

$$j \circ L_v j = L_{\mathbf{j}v} j$$

As \hat{f} is holomorphic, $Jdf(v) = d\hat{f}(\mathbf{j}v)$. In particular, the map $v \mapsto (L_v j, df(v))$ is complex, so the map $T_f \mathbf{F}(\hat{f}) \longrightarrow T_f \mathcal{M}^{st}$ is complex linear. \square

The canonical homotopy class of complex structure on $T_f \mathbf{F}(\hat{f}) \downarrow_{\mathbf{B}_0}$ gives a canonical homotopy class of complex structure on $T\mathbf{F}(\hat{f}) \downarrow_{\mathbf{B}_0}$ in a neighborhood of f . To verify that this agrees with the canonical homotopy class of complex structure given by other holomorphic curves in this neighborhood, we shall extend the definition of $K_{i,t}$ to all curves in a neighborhood of f in \hat{f} . As a first step, we shall modify V_i off \hat{f}_i to make later application of Theorem 3.8 easier.

Lemma 8.2. *After restricting \hat{f}_i to a neighborhood of f if necessary, there exists a core family $(\hat{f}/G, \{s_i\})$ with*

- *an identification of G with G_i and a G -equivariant embedding*

$$\hat{f}_i \longrightarrow \hat{f}$$

so that for $t \in [0, 1]$ there is no nonzero section of $f^ T_{vert} \hat{\mathbf{B}}$ in $((1-t)D\bar{\partial} + tD\bar{\partial}^C)^{-1} V_i(f)$ which also vanishes on the image of all the sections $\{s_i\}$,*

- *and a locally free, G -invariant subsheaf V' of $\Gamma^{(0,1)}(T_{vert}^* \mathbf{C}(\hat{f}) \otimes T_{vert} \hat{\mathbf{B}})$ with the same dimension as V_i , and which pulls back to \hat{f}_i (in the sense of Definition 2.19) to give $V_i(\hat{f}_i)$.*

Proof:

After restricting \hat{f}_i to a neighborhood of any holomorphic curve f in \hat{f}_i , Claim 6.10 from the proof of Theorem 6.6 provides an equivariant, fiberwise holomorphic embedding

$$\begin{array}{ccc} \mathbf{C}(\hat{f}_i) & \longrightarrow & \mathbf{C}(\hat{f}_0) \\ \downarrow & & \downarrow \\ \mathbf{F}(\hat{f}_i) & \longrightarrow & \mathbf{F}(\hat{f}_0) \end{array}$$

where, as specified by Lemma 6.4 part 3, $(\hat{f}_0/G, \{s_i\})$ is a core family containing f with enough sections $\{s_i\}$ that $((1-t)D\bar{\partial} + tD\bar{\partial}^C)^{-1} V(f)$ contains no nonzero sections of $f^* T_{vert} \hat{\mathbf{B}}$ which vanish on the image of all $\{s_i\}$. The above map removes the G -fold ambiguity of the core family map $\mathbf{C}(\hat{f}_i) \longrightarrow \mathbf{C}(\hat{f}_0)/G$, so \hat{f}_i is equal to the above map composed with \hat{f}_0 followed by exponentiation of some section of $\hat{f}_0^* T_{vert} \hat{\mathbf{B}}$ which vanishes on the image of the sections $\{s_i\}$. We may choose this section to be G -equivariant, and define \hat{f} to be f_0 followed by exponentiation of this section. Now the above map corresponds to a G -equivariant embedding

$$\hat{f}_i \longrightarrow \hat{f}$$

and $(\hat{f}/G, \{s_i\})$ is a core family satisfying the requirements of this lemma. In particular, as $\hat{f}_i \longrightarrow \hat{f}$ is an embedding, we may choose a locally free subsheaf V' of $\Gamma^{(0,1)}(T_{vert}^* \mathbf{C}(\hat{f}) \otimes T_{vert} \hat{\mathbf{B}})$ which pulls back in the sense of Definition 2.19 to give $V_i(\hat{f}_i)$, and which has the same rank as V_i , when considered as a sheaf of $C^\infty \perp(\mathbf{F}(\hat{f}))$ modules. We may choose V' to be G -invariant. \square

The pullback of V' defines some subsheaf V of \mathcal{Y} on a neighborhood of \hat{f}_i . For any family of curves \hat{h} in this neighborhood, the subset of curves h in \hat{h} for which the dimension of $V(h)$ is equal to the rank of V' is open, and contains all such

curves h isomorphic to f . Lemma 2.8 then implies that on a neighborhood of f , V is simply generated in the sense of Definition 2.20.

As $V(f) = V_i(f)$, $\bar{\partial}$ is strongly transverse to $V(f)$, so we may apply Theorem 6.6 to see that in a neighborhood of f the moduli stack of curves with $\bar{\partial}$ in V is represented the quotient of a family of curves by a group of automorphisms. Theorem 6.6 also implies that this family of curves has the same dimension as \hat{f}_i , therefore it follows that this moduli stack of curves with $\bar{\partial}$ in V is the same as the moduli stack of curves with $\bar{\partial}$ in V_i , which is represented by \hat{f}_i/G (restricted to a neighborhood of f if necessary).

We shall now continue to define an extension of $K_{i,t}(f)$ using V instead of V_i .

To linearize $\bar{\partial}$ at a non-holomorphic curve, we need a connection on \mathcal{Y} . Instead, we shall linearize $\pi_V \bar{\partial}$, where π_V is the projection

$$\pi_V : \mathcal{Y} \longrightarrow \mathcal{Y}/V$$

For any curve f so that $\bar{\partial}f \in V(f)$, we may define the linear map

$$D\pi_V \bar{\partial} : T_f \mathcal{M}^{st} \longrightarrow \mathcal{Y}(f)/V(f)$$

as follows: In light of Lemma 2.13, we need only construct $D\pi_V \bar{\partial}$ on $T_f \mathbf{F}(\hat{f})$ for an arbitrary $C^{\infty,1}$ family \hat{f} containing f . So long as this construction commutes with maps of families and gives a linear map, Lemma 2.13 implies that it shall define a linear map from $T_f \mathcal{M}^{st}$. Choose a section v of $V(\hat{f})$ equal to $\bar{\partial}f$ at f in \hat{f} . Then $\bar{\partial}f_t - v$ is a section of $\mathcal{Y}(\hat{f})$ which vanishes at f , therefore its derivative at f is a linear map

$$T_f \mathbf{F}(\hat{f}) \longrightarrow \mathcal{Y}(f)$$

Any other choice of v would change the above map by an linear map to $V(f)$. This derivative therefore gives a well defined linear map

$$D\pi_V \bar{\partial} : T_f \mathbf{F}(\hat{f}) \longrightarrow \mathcal{Y}(f)/V(f)$$

As the above construction is compatible with maps of families, Lemma 2.13 implies that this construction gives a well defined linear map $T_f \mathcal{M}^{st} \longrightarrow \mathcal{Y}(f)/V(f)$.

Lemma 8.3. *On a neighborhood U of all holomorphic curves in \hat{f}_i , there exists a map (of sheaves $C^{\infty,1}(\mathbf{F}(\hat{f}_i))$ modules over $\mathbf{F}(\hat{f}_i)$)*

$$D\pi_V \bar{\partial} : T_{\hat{f}_i} \mathcal{M}^{st} \longrightarrow \mathcal{Y}(\hat{f}_i)/V(\hat{f}_i)$$

which restricts to each curve f' in U to be $D\pi_V \bar{\partial} : T_{f'} \mathcal{M}^{st} \longrightarrow \mathcal{Y}(f')/V(f')$.

Proof:

If such a map exists it is uniquely determined by its restriction to $T_{f'} \mathcal{M}^{st}$ for each f' in U . We therefore need only construct such a map in a neighborhood of a holomorphic curve f in \hat{f}_i .

To globalize the definition of $D\pi_V \bar{\partial}$, note that there exists a neighborhood of f in \mathcal{M}^{st} with a global section θ of the sheaf V so that $\theta(\hat{f}_i) = \bar{\partial}\hat{f}_i$. In particular, as \hat{f}_i embeds into \hat{f} , the section $\bar{\partial}$ of \hat{f}_i is the pullback of some section θ' of V' over $\mathbf{F}(\hat{f})$. As $\bar{\partial}$ is G -invariant and the inclusion $\hat{f}_i \longrightarrow \hat{f}$ is G -equivariant, averaging allows us to construct θ' to be G -invariant. The pullback of θ' to a neighborhood of f defines a section θ of $V \subset \mathcal{Y}$.

Given a section v of $T_{\hat{f}_i} \mathcal{M}^{st}$, Lemma 2.17 implies that there is a one dimensional deformation \hat{f}_t of \hat{f}_i so that the derivative of \hat{f}_t at $t = 0$ is v . The derivative of $\bar{\partial}\hat{f}_t - \theta(\hat{f}_t)$ at $t = 0$ is a section of $\mathcal{Y}(\hat{f}_i)$, which when restricted to any curve f' in

\hat{f}_i and followed by projection to $\mathcal{Y}(f')/V(f')$ is equal to $D\pi_V \bar{\partial}$ of v restricted to f' . We have just proven that given any $C^{\infty,1}$ section v of $T_{\hat{f}_i} \mathcal{M}^{st}$, there exists a $C^{\infty,1}$ section $D\bar{\partial} \pi_V v$ of $\mathcal{Y}(\hat{f}_i)$ which restricts to each $\mathcal{Y}(f')$ to be equal to $D\pi_V \bar{\partial}(v(f'))$. As this characterization uniquely determines $D\pi_V \bar{\partial} v$ It follows that

$$D\pi_V \bar{\partial} : T_{\hat{f}_i} \mathcal{M}^{st} \longrightarrow \mathcal{Y}(\hat{f}_i)/V(\hat{f}_i)$$

is a well defined map of sheaves of $C^{\infty,1}$ modules over $\mathbf{F}(\hat{f}_i)$. \square

$\mathcal{Y}(\hat{f}_i)/V(\hat{f}_i)$ has a canonical complex structure because on \hat{f}_i , V coincides with V_i , which is complex. In order to talk of the complex linear part $(D\pi_V \bar{\partial})^{\mathbb{C}}$ of $D\pi_V \bar{\partial}$ restricted to $T_{\hat{f}_i} \mathcal{M}^{st} \downarrow_{\mathbf{B}_0}$, we shall need to choose a complex structure on $T_{\hat{f}_i} \mathcal{M}^{st} \downarrow_{\mathbf{B}_0}$.

Claim 8.4. *There exists a G_i -invariant complex structure J' on $T_{\hat{f}_i} \mathcal{M}^{st} \downarrow_{\mathbf{B}_0}$ defined within a neighborhood U of f so that*

- *the restriction of this complex structure J' to any holomorphic curve f' in this neighborhood U is the canonical complex structure on $T_{f'} \mathcal{M}^{st} \downarrow_{\mathbf{B}_0}$,*
- *if $X(\hat{f}_i)$ indicates the sheaf of $C^{\infty,1}$ sections of $\hat{f}_i^* T_{vert} \hat{\mathbf{B}}$ which vanish on the image of the sections $\{s_i\}$ from the core family $(\hat{f}/G, \{s_i\})$ from Lemma 8.2, the inclusion*

$$X(\hat{f}_i) \longrightarrow T_{\hat{f}_i} \mathcal{M}^{st} \downarrow_{\mathbf{B}_0}$$

is complex.

Proof:

Recall that the sheaf $T_{\hat{f}_i} \mathcal{M}^{st} \downarrow_{\mathbf{B}_0}$ is defined using the short exact sequence

$$0 \rightarrow \Gamma(T_{vert} \mathbf{C}(\hat{f}_i)) \rightarrow \Gamma^{0,1}(T_{vert} \mathbf{C}(\hat{f}_i) \otimes T_{vert}^* \mathbf{C}(\hat{f}_i)) \times \Gamma(\hat{f}_i^* T_{vert} \hat{\mathbf{B}}) \rightarrow T_{\hat{f}_i} \mathcal{M}^{st}(\hat{\mathbf{B}}) \downarrow_{\mathbf{B}_0} \rightarrow 0$$

The sheaf $\Gamma^{0,1}(T_{vert} \mathbf{C}(\hat{f}_i) \otimes T_{vert}^* \mathbf{C}(\hat{f}_i)) \times \Gamma(\hat{f}_i^* T_{vert} \hat{\mathbf{B}})$ has a complex structure, but $T_{\hat{f}_i} \mathcal{M}^{st} \downarrow_{\mathbf{B}_0}$ fails to have a canonical complex structure because the left hand map above is not in general complex. Below we shall construct a complex structure on $T_{\hat{f}_i} \mathcal{M}^{st} \downarrow_{\mathbf{B}_0}$ by splitting the above exact sequence with an inclusion of $T_{\hat{f}_i} \mathcal{M}^{st} \downarrow_{\mathbf{B}_0}$ as a complex subsheaf of $\Gamma^{0,1}(T_{vert} \mathbf{C}(\hat{f}_i) \otimes T_{vert}^* \mathbf{C}(\hat{f}_i)) \times \Gamma(\hat{f}_i^* T_{vert} \hat{\mathbf{B}})$.

Recall that we have \hat{f}_i embedded in a core family \hat{f} , and that $X(\hat{f}_i)$ indicates the sheaf of $C^{\infty,1}$ sections of $\hat{f}_i^* T_{vert} \hat{\mathbf{B}}$ which vanish on the sections $\{s_i\}$ of the core family $(\hat{f}/G, \{s_i\})$. There is a canonical injective map $X(\hat{f}_i) \longrightarrow T_{\hat{f}_i} \mathcal{M}^{st} \downarrow_{\mathbf{B}_0}$ which has cokernel which may be identified with the pullback of $T\mathbf{F}(\hat{f}) \downarrow_{\mathbf{B}_0}$ to $\mathbf{F}(\hat{f}_i)$.

Note that $X(\hat{f}_i)$ is a complex subsheaf of $\Gamma^{0,1}(T_{vert} \mathbf{C}(\hat{f}_i) \otimes T_{vert}^* \mathbf{C}(\hat{f}_i)) \times \Gamma(\hat{f}_i^* T_{vert} \hat{\mathbf{B}})$. At our holomorphic curve f , the inclusion $X(f) \longrightarrow T_f \mathcal{M}^{st} \downarrow_{\mathbf{B}_0}$ is complex, and has finite dimensional cokernel. There exists a locally free, finite rank, complex, G_i -invariant subsheaf W of $\Gamma^{0,1}(T_{vert} \mathbf{C}(\hat{f}_i) \otimes T_{vert}^* \mathbf{C}(\hat{f}_i)) \times \Gamma(\hat{f}_i^* T_{vert} \hat{\mathbf{B}})$ which restricted to f is a finite dimensional vector space which is complementary to the direct sum of the image of $\Gamma(T\mathbf{C}(f))$ with $X(f)$.

Restricted to any other curve f' , W is complementary to direct sum of the image of $\Gamma(T\mathbf{C}(f'))$ with $X(f')$ if and only if its image in $T_{f'} \mathbf{F}(\hat{f}) \downarrow_{\mathbf{B}_0}$ is surjective. This holds for all f' in a neighborhood of f in \hat{f}_i . On this neighborhood the map $X(\hat{f}_i) \oplus W \longrightarrow T_{\hat{f}_i} \mathcal{M}^{st} \downarrow_{\mathbf{B}_0}$ is an isomorphism, so we may use the complex structure from $X(\hat{f}_i) \oplus W$ to give a G_i -invariant complex structure on $T_{\hat{f}_i} \mathcal{M}^{st} \downarrow_{\mathbf{B}_0}$ in a neighborhood of f . For f' holomorphic, this complex structure on $T_{f'} \mathcal{M}^{st} \downarrow_{\mathbf{B}_0}$ agrees with the canonically defined one. \square

Remark 8.5. In the proof of Claim 8.4 above, we considered $T_{\hat{f}_i} \mathcal{M}^{st} \downarrow_{\mathbf{B}_0}$ as embedded inside $\Gamma^{0,1}(T_{vert} \mathbf{C}(\hat{f}_i) \otimes T_{vert}^* \mathbf{C}(\hat{f}_i)) \times \Gamma(\hat{f}_i^* T_{vert} \hat{\mathbf{B}})$. We may use this embedding to construct a metric on $T_{\hat{f}_i} \mathcal{M}^{st} \downarrow_{\mathbf{B}_0}$. Our embedding represents sections of $T_{\hat{f}_i} \mathcal{M}^{st} \downarrow_{\mathbf{B}_0}$ as sections of some vector bundle over $\mathbf{C}(\hat{f}_i)$. We may choose a $C^{\infty,1}$ inner product $\langle \cdot, \cdot \rangle_0$ on this vector bundle, and a $C^{\infty,1}$ fiberwise volume form θ on $\pi : \mathbf{C}(\hat{f}_i) \rightarrow \mathbf{F}(\hat{f}_i)$ to define an inner product $\langle \cdot, \cdot \rangle$ on $T_{\hat{f}_i} \mathcal{M}^{st} \downarrow_{\mathbf{B}_0}$ as

$$\langle v, w \rangle := \pi! \langle v, w \rangle_0 \theta$$

so $\langle v, w \rangle$ is a $C^{\infty,1}$ function on $\mathbf{F}(\hat{f}_i)$ with value at a point p given by integration of $\langle v, w \rangle_0 \theta$ over the fiber of $\mathbf{C}(\hat{f}_i)$ over p . (The fact that $\langle v, w \rangle$ is $C^{\infty,1}$ follows from the construction of $\pi!$ in [12].)

With our chosen complex structure on $T_{\hat{f}_i} \mathcal{M}^{st} \downarrow_{\mathbf{B}_0}$, we may define the complex linear part, $(D\pi_V \bar{\partial})^{\mathbb{C}}$ of $D\pi_V \bar{\partial}$. Define

$$A_t := (1-t)D\pi_V \bar{\partial} + t(D\pi_V \bar{\partial})^{\mathbb{C}} : T_{\hat{f}_i} \mathcal{M}^{st} \downarrow_{\mathbf{B}_0} \rightarrow \mathcal{Y}(\hat{f}_i)/V(\hat{f}_i)$$

Lemma 8.6. On a neighborhood of f in \hat{f}_i ,

$$\ker A_t \subset T_{\hat{f}_i} \mathcal{M}^{st} \downarrow_{\mathbf{B}_0}$$

is a one dimensional family of finite dimensional sub vector bundles of $T_{\hat{f}_i} \mathcal{M}^{st} \downarrow_{\mathbf{B}_0}$ (in other words $\ker A_t$ is a one dimensional family of subsheaves of $T_{\hat{f}_i} \mathcal{M}^{st}$ which are free and finite rank sheaves of $C^{\infty,1}(\mathbf{F}(\hat{f}_i))$ modules.)

Proof:

We shall be using the notation from the proof of Lemma 8.3. Within the proof of Lemma 8.3, we constructed $D\pi_V \bar{\partial}$ as the linearization of $\bar{\partial} - \theta$ followed by projection to \mathcal{Y}/V . The section θ of V is the pullback of a section θ' of V' over $\mathbf{F}(\hat{f})$. As \hat{f}_i is embedded as a sub family of \hat{f} , may restrict θ' to $\mathbf{F}(\hat{f}_i)$ (and again call it θ').

Let $X(\hat{f}_i)$ denote the sheaf of $C^{\infty,1}$ sections of $\hat{f}_i^* T_{vert} \hat{\mathbf{B}}$ which vanish on the pullback of the sections $\{s_i\}$ from the core family \hat{f} . After choosing a G -invariant trivialization for \hat{f}_i in the sense of Definition 3.1, we may use θ' to get a simple perturbation of $\bar{\partial} : X(\hat{f}_i) \rightarrow \mathcal{Y}(\hat{f}_i)$ which may be thought of as

$$\bar{\partial}' := \bar{\partial} - \theta'$$

but is more accurately described in Example 3.5. We may now apply Theorem 3.8 to the linearization of $\bar{\partial}'$. At the curve f , the linearization of $\bar{\partial}$ is equal to the linearization of $\bar{\partial}'$ up to a map to $V(f)$. As specified by Lemma 8.2, $((1-t)D\bar{\partial} + tD\bar{\partial}^{\mathbb{C}})^{-1}(V(f))$ does not contain any nonzero elements of $X(f)$. It follows that

$$((1-t)D\bar{\partial}' + tD\bar{\partial}'^{\mathbb{C}}) : X(f) \rightarrow \mathcal{Y}(f)$$

is injective for all $t \in [0, 1]$. Remark 3.4 of [13] implies that Theorem 3.8 applies to the above homotopy. Theorem 3.8 part 3 then implies that for all curves f' in some neighborhood of f in \hat{f}_i ,

$$((1-t)D\bar{\partial}' + tD\bar{\partial}'^{\mathbb{C}}) : X(f') \rightarrow \mathcal{Y}(f')$$

is injective. Theorem 3.8 part 4 implies that on the neighborhood where the above maps are injective, the cokernel,

$$E_t(\hat{f}_i) := \mathcal{Y}(\hat{f}_i) / ((1-t)D\bar{\partial}' + tD\bar{\partial}'^{\mathbb{C}})(X(\hat{f}_i))$$

is a finite dimensional $C^{\infty,1}$ vector bundle.

At f , the natural map $V(f) \rightarrow E_t(f)$ is injective, therefore on some neighborhood of f , the natural map of vector bundles $V(\hat{f}_i) \rightarrow E_t(\hat{f}_i)$ is injective, so the quotient of $E_t(\hat{f}_i)$ by the image of $V(\hat{f}_i)$ is a finite dimensional, $C^{\infty,1}$ vector

bundle, $E'_t(\hat{f}_i)$. As $X(\hat{f}_i)$ is a complex subsheaf of $T_{\hat{f}_i}\mathcal{M}^{st}$ with the complex structure chosen in Claim 8.4, the restriction of A_t to $X(\hat{f}_i)$ is equal to the following composition

$$\begin{array}{ccc} X(\hat{f}_i) & \xrightarrow{((1-t)D\bar{\partial}' + tD\bar{\partial}'^c)} & \mathcal{Y}(\hat{f}_i) \xrightarrow{\pi_V} \mathcal{Y}(\hat{f}_i)/V(\hat{f}_i) \\ & \searrow A_t & \nearrow \end{array}$$

and the cokernel of A_t restricted to $X(\hat{f}_i)$ is equal to E'_t , so the following is a short exact sequence

$$0 \longrightarrow X(\hat{f}_i) \xrightarrow{A_t} \mathcal{Y}(\hat{f}_i)/V(\hat{f}_i) \longrightarrow E'_t(\hat{f}_i) \longrightarrow 0$$

As A_t is defined on all of $T_{\hat{f}_i}\mathcal{M}^{st}\downarrow_{\mathbf{B}_0}$, it induces a $C^{\infty,1}$ map of finite dimensional vector bundles

$$A'_t : (T_{\hat{f}_i}\mathcal{M}^{st}\downarrow_{\mathbf{B}_0})/X(\hat{f}_i) \longrightarrow E'_t(\hat{f}_i)$$

The assumption that $\bar{\partial}$ is strongly transverse to V at f implies that A'_t is surjective at f , and is therefore surjective on a neighborhood of f . On this neighborhood, $\ker A'_t$ is a finite dimensional sub-vectorbundle of $(T_{\hat{f}_i}\mathcal{M}^{st}\downarrow_{\mathbf{B}_0})/X(\hat{f}_i)$.

Any $C^{\infty,1}$ section of the kernel of A'_t lifts to a $C^{\infty,1}$ section v of $T_{\hat{f}_i}\mathcal{M}^{st}\downarrow_{\mathbf{B}_0}$ so that $A_t(v)$ is in $A_t(X(\hat{f}))$. As A_t is injective restricted to $X(\hat{f})$, it follows that there is a unique $C^{\infty,1}$ section x of $X(\hat{f})$ so that $v - x$ is in the kernel of A_t . In other words, any $C^{\infty,1}$ section of the kernel of A'_t lifts uniquely to a $C^{\infty,1}$ section of the kernel of A_t . It follows that $\ker A_t$ is a finite dimensional $C^{\infty,1}$ sub vectorbundle of $T_{\hat{f}_i}\mathcal{M}^{st}\downarrow_{\mathbf{B}_0}$. □

Observe that for any holomorphic curve f' in \hat{f}_i , the complex structure on $T_{f'}\mathcal{M}^{st}\downarrow_{\mathbf{B}_0}$ chosen in Claim 8.4 agrees with the canonically defined complex structure, so the restriction of $\ker A_t$ to f' is equal to $K_{i,t}(f')$. The kernel of A_t is therefore our required extension of $K_{i,t}$ to a family of vector bundles defined on all of \hat{f}_i . Note also that as A_1 is complex, $\ker A_1$ has a complex structure which agrees with the complex structure on $K_{i,1}(f')$.

As our choice of complex structure on $T_{\hat{f}_i}\mathcal{M}^{st}\downarrow_{\mathbf{B}_0}$ was G_i -invariant, and $D\pi_V\bar{\partial}$ is intrinsically defined, $\ker A_t$ is a G_i -invariant family of sub vectorbundles of $T_{\hat{f}_i}\mathcal{M}^{st}\downarrow_{\mathbf{B}_0}$. We therefore have a canonical class of G_i -invariant complex structures on $\ker A_0$, which restricts at any holomorphic curve f' to the canonical class of complex structures on $K_{i,0}(f')$ which is invariant under the stabilizer of $[f']$.

Lemma 8.7. *In a neighborhood of the holomorphic curves in \hat{f}_i , $\ker A_0$ is equal to $T\mathbf{F}(\hat{f}_i)\downarrow_{\mathbf{B}_0}$.*

Proof: At holomorphic curves f' , Theorem 6.6 implies that $K_{i,0}(f')$ and $\ker A_0(f')$ are both equal to $T_{f'}\mathbf{F}(\hat{f}_i)\downarrow_{\mathbf{B}_0}$.

More generally, the fact that $\bar{\partial}\hat{f}_i$ is a section of $V(\hat{f}_i)$ implies that the map $T\mathbf{F}(\hat{f}_i)\downarrow_{\mathbf{B}_0} \longrightarrow T_{\hat{f}_i}\mathcal{M}^{st}\downarrow_{\mathbf{B}_0}$ has image inside $\ker A_0 = \ker D\pi_V\bar{\partial}$. As noted above, this $C^{\infty,1}$ map of finite dimensional vector bundles $T\mathbf{F}(\hat{f}_i)\downarrow_G \longrightarrow \ker A_0$ is an isomorphism restricted to holomorphic curves f' . Therefore, it is an isomorphism in a neighborhood of these holomorphic curves. □

We have now locally constructed an extension of $K_{i,t}$ to a family of G_i -equivariant vector sub bundles $\ker A_t \subset T_{\hat{f}_i}\mathcal{M}^{st}\downarrow_{\mathbf{B}_0}$. More generally, say $K'_{i,t}$ is an extension

of $K_{i,t}$ if it is a family of G_i -equivariant vector sub bundles of $T_{\hat{f}_i} \mathcal{M}^{st} \downarrow_{\mathbf{B}_0}$ so that for any holomorphic curve f in the domain of definition, $K'_{i,t}(f) = K_{i,t}(f)$.

We may think of $K_{i,t}$ as a family of G_i -equivariant vector bundles over $\{\bar{\partial} \hat{f}_i = 0\}$. In particular, we may define a $C^{\infty,1}$ section of $K_{i,t}$ to be a section which extends to a $C^{\infty,1}$ section of $T_{\hat{f}_i} \mathcal{M}^{st} \downarrow_{\mathbf{B}_0}$. The following lemma implies that any such $C^{\infty,1}$ section of $K_{i,t}$ extends to a $C^{\infty,1}$ section of any extension $K'_{i,t}$ of $K_{i,t}$.

Lemma 8.8. *Any section of $K_{i,t}$ which extends to a $C^{\infty,1}$ section of $T_{\hat{f}_i} \mathcal{M}^{st} \downarrow_{\mathbf{B}_0}$ also extends to a $C^{\infty,1}$ section of any extension $K'_{i,t}$ of $K_{i,t}$.*

Proof: We may prove this using the metric on $T_{\hat{f}_i} \mathcal{M}^{st} \downarrow_{\mathbf{B}_0}$ constructed in Remark 8.5. Given any $C^{\infty,1}$ section of $T_{\hat{f}_i} \mathcal{M}^{st} \downarrow_{\mathbf{B}_0}$, its orthogonal projection to $K'_{i,t}$ is a $C^{\infty,1}$ section of $K'_{i,t}$. In particular, any $C^{\infty,1}$ extension of a section of $K_{i,t}$ to $T_{\hat{f}_i} \mathcal{M}^{st} \downarrow_{\mathbf{B}_0}$ projects to a $C^{\infty,1}$ extension contained within $K'_{i,t}$. \square

Lemma 8.8 implies that the $C^{\infty,1}$ vector bundle structure on $K_{i,t}$ induced by including $K_{i,t}$ inside $K'_{i,t}$ does not depend on which extension $K'_{i,t}$ of $K_{i,t}$ is chosen.

Definition 8.9. *A t -trivialization of $K_{i,t}$ is a choice of G_i -equivariant identification of $K_{i,t}$ with $K_{i,0}$ for all t which extends to a $C^{\infty,1}$ family of isomorphisms $K'_{i,t} \rightarrow K'_{i,0}$ and which is the identity on the subspace of \mathbb{R} -nil vectors within $K_{i,t}$.*

Given a submersion $\Phi : \mathcal{M}^{st} \rightarrow \mathbf{X}$, say that a t -trivialization is Φ -submersive if the following diagram commutes

$$\begin{array}{ccc} K_{i,t}(f) & \xrightarrow{T_f \Phi} & T\mathbf{X} \\ \downarrow & \nearrow T_f \Phi & \\ K_{i,0}(f) & & \end{array}$$

A compatible choice of t -trivializations for an embedded Kuranishi structure, $\{(U_i, V_i, \hat{f}_i/G_i)\}$, is a choice of t -trivialization for $K_{i,t}$ on $(U_i^\sharp, V_i, \hat{f}_i^\sharp/G_i)$ for all i so that whenever there is an inclusion $K_{i,t}(f) \rightarrow K_{j,t}(f)$, the following diagram commutes

$$\begin{array}{ccccc} K_{i,t}(f) & \longrightarrow & K_{j,t}(f) & \xrightarrow{(1-t)D\pi_{V_i} \bar{\partial} + t(D\pi_{V_i} \bar{\partial})^c} & V_j/V_i \\ \downarrow & & \downarrow & \nearrow D\pi_{V_i} \bar{\partial} & \\ K_{i,0}(f) & \longrightarrow & K_{j,0}(f) & & \end{array}$$

Note that Lemma 8.8 implies that the definition of a t -trivialization does not depend on which extension $K'_{i,t}$ of $K_{i,t}$ is chosen.

We shall be interested in Φ -submersive t -trivializations in the case that Φ is holomorphic in the sense of Definition 2.24 on page 19. The following proposition constructs compatible Φ -submersive t -trivializations. Most constructions of compatible objects on Kuranishi structures use transfinite induction in very similar arguments to the following one.

Proposition 8.10. *Given a holomorphic submersion $\Phi : \mathcal{M}^{st} \rightarrow \mathbf{X}$, and a choice of Φ -submersive embedded Kuranishi structure $\{(U_i, V_i, \hat{f}_i/G_i)\}$ on $\mathcal{M} \subset \mathcal{M}^{st}$, there exists a compatible choice of Φ -submersive, t -trivializations of $K_{i,t}$.*

Given any such choice defined on a neighborhood of a closed substack $\mathcal{M}' \subset \mathcal{M}$, our choice may be made to coincide with the original choice when restricted to a (possibly smaller) neighborhood of \mathcal{M}' .

Proof: The construction of a t -trivialization shall proceed by transfinite induction. In particular, we shall choose a well ordering of our Kuranishi charts, then construct our t -trivialization in this order. At each step we shall need to shrink the domain of definition a little, so we shall also need to specify more than one extension of each Kuranishi chart.

Claim 8.11. *There is a well-ordering \prec of the Kuranishi charts so that $j \prec i$ if $\dim V_j < \dim V_i$, and so that for any fixed i , $j \prec i$ for only a finite number of j with $\dim V_j = \dim V_i$.*

There are compatible extensions $(\mathcal{U}_{i,k}, V_i, \hat{f}_{i,k}/G_i)$ and $(\mathcal{U}_{i,\underline{k}}, V_i, \hat{f}_{i,\underline{k}}/G_i)$ of $(\mathcal{U}_i, V_i, \hat{f}_i/G_i)$ for all $k \succeq i$ so that

- *if $k' \prec k$, $\hat{f}_{i,k'}$ is an extension of $\hat{f}_{i,k}$ and is contained in $\hat{f}_{i,\underline{k}}$*
- *$\hat{f}_{i,\underline{k}}$ is an extension of $\hat{f}_{i,k}$*
- *and the intersection of $\hat{f}_{i,k}$ for all $k \succeq i$ contains an extension of $\hat{f}_{i,k}$.*

The existence of such an order on our Kuranishi charts is obvious, because there are only a countable number of charts. To construct $\mathcal{U}_{i,k}$, note that by definition $(\mathcal{U}_i, V_i, \hat{f}_i/G_i)$ is extendible, so there exists an extension $(\mathcal{U}_i^\sharp, V_i, \hat{f}_i^\sharp/G_i)$ and a $C^{\infty,1}$ map $\rho : \mathcal{U}_i^\sharp \rightarrow (0, 1]$ satisfying the requirements of Definition 2.27 so that \mathcal{U}_i is the substack where $\rho > 1/2$. We may embed our well ordered set of indices into $(0, 3/8)$ as follows:

$$x_k := 3/8 - 2^{-\dim V_k - 2}(1 + 1/(n_k + 2)) \in (0, 3/8)$$

where n_k is the number of indices $j \preceq k$ with $\dim V_j = \dim V_k$. The only important property of x_k is that

$$x_k > \sup_{j \prec k} x_j$$

We may then define

$$\mathcal{U}_{i,k} := \{\rho > x_k\} \subset \mathcal{U}_i^\sharp$$

and

$$\mathcal{U}_{i,\underline{k}} := \{\rho > \sup_{j \prec k} x_j\} \subset \mathcal{U}_i^\sharp$$

These open substacks $\mathcal{U}_{i,k}$ satisfy all the requirements of Claim 8.11 above.

Compatibility between $K_{j,t}$ and $K_{i,t}$ for $j \prec i$ will only be required on the intersection of $\mathcal{U}_{j,i}$ with $\mathcal{U}_{i,i}$. Similarly, choose some open neighborhood O of $\mathcal{M}' \subset \mathcal{M}$ so that the closure of O is contained in the neighborhood of \mathcal{M}' on which a t -trivialization is already chosen. Our new t -trivialization shall only be required to agree with the already chosen t -trivialization on O .

Let f be a holomorphic curve in $\hat{f}_{i,i}$. Suppose that we have a compatible choice of t -trivialization for all $K_{j,t}$ on $\hat{f}_{j,k}$ for $j \preceq k \prec i$. We shall construct a t -trivialization of $K_{i,t}$ in a neighborhood of f in three different cases below:

- (1) If f is contained in \mathcal{M}' , then there is already a chosen t -trivialization of $K_{i,t}$ which we shall assume compatible with our trivializations of $K_{j,t}$ for $j \prec i$ by construction.
- (2) If f is contained in $\mathcal{U}_{j,\underline{i}}$ for some $j \prec i$ but not in \mathcal{M}' , proceed as follows: Without losing generality, we may assume that V_j has the largest dimension so that f is in $\mathcal{U}_{j,\underline{i}}$. We shall construct a t -trivialization of $K_{i,t}$ on some open subset of $\hat{f}_{i,i} \cap \mathcal{U}_{j,\underline{i}}$ where we already have a trivialization of $K_{j,t} \subset K_{i,t}$.

On a neighborhood of f , construct an extension $K'_{i,t}$ of $K_{i,t}$ using the complex structure on $T_{\hat{f}_{i,i}} \mathcal{M}^{st} \downarrow_{\mathbf{B}_0}$ from Claim 8.4 to construct a homotopy of $D\pi_{V_i} \bar{\partial}$ to its complex linear part, and letting $K'_{i,t}$ be the kernel.

Because this complex structure is G_i -invariant, and there is a unique map $\hat{f}_{j,\underline{i}} \rightarrow \hat{f}_{i,i}/G_i$ (defined in a neighborhood of f), we may pull this complex structure back to a complex structure on $T_{\hat{f}_{j,\underline{i}}} \mathcal{M}^{st} \downarrow_{\mathbf{B}_0}$. Using this pulled back complex structure, we may construct an extension $K'_{j,t}$ of $K_{j,t}$ on a neighborhood of f in $\hat{f}_{j,\underline{i}}$ with the property that $K'_{j,t}(f') \subset K'_{i,t}(f')$. In particular, construct $K'_{j,t}$ as the kernel of the operator

$$A_t := (1-t)D\pi_{V_j}\bar{\partial} + t(D\pi_{V_j}\bar{\partial})^{\mathbb{C}} : T_{\hat{f}_{j,\underline{i}}} \mathcal{M}^{st} \downarrow_{\mathbf{B}_0} \rightarrow \mathcal{Y}(\hat{f}_{j,\underline{i}})/V_j$$

$K'_{i,t}$ is defined using the analogous operator with V_i in place of V_j . As $V_j \subset V_i$, and the same complex structure is used to define both $K'_{i,t}(f')$ and $K'_{j,t}(f')$, A_t sends $K'_{i,t}(f')$ to V_i/V_j .

As our embedded Kuranishi structure is Φ -submersive, A_t restricted to the kernel of $T_f\Phi$ is surjective, $T_f\Phi : K'_{j,t}(f) \rightarrow T_{\Phi(f)}\mathbf{X} \downarrow_{\mathbf{X}_0}$ is surjective, and the same holds for all f' in some neighborhood of f . It follows that

$$K'_{i,t}(f')/K'_{j,t}(f') \xrightarrow{A_t} V_i/V_j$$

and

$$(\ker T_f\Phi \cap K'_{i,t}(f')) / (\ker T_f\Phi \cap K'_{j,t}(f')) \xrightarrow{A_t} V_i/V_j$$

are both isomorphisms for f' in a neighborhood of f .

Use the notation $(K'_{j,t})^{\perp}(f')$ to denote the orthogonal complement of $\ker T_f\Phi \cap K'_{j,t}(f')$ within $\ker T_f\Phi \cap K'_{i,t}(f')$. (Use the equivariant metric from Remark 8.5.) We may split $K'_{i,t}(f')$ into $K'_{j,t}(f') \oplus (K'_{j,t})^{\perp}(f')$. There is a G_j -equivariant sub vector bundle $(K'_{j,t})^{\perp}$ of $T_{\hat{f}_{j,\underline{i}}} \mathcal{M}^{st} \downarrow_{\mathbf{B}_0}$ which restricts to f' to be $(K'_{j,t})^{\perp}(f')$. On a neighborhood of f in $\hat{f}_{j,\underline{i}}$, A_t defines an isomorphism of vector bundles

$$A_t : (K'_{j,t})^{\perp} \rightarrow V_i/V_j$$

Our t -trivialization of $K_{j,t}$ extends by definition to a t -trivialization of $K'_{j,t}$, which we may take to be G_j -equivariant. There is a canonical G_j -equivariant t -trivialization of $(K'_{j,t})^{\perp}$ so that the diagram

$$\begin{array}{ccc} (K'_{j,t})^{\perp} & \xrightarrow{A_t} & V_i/V_j \\ \uparrow & \nearrow A_0 & \\ (K'_{j,0})^{\perp} & & \end{array}$$

commutes. The corresponding G_j -equivariant t -trivialization of $K'_{j,t} \oplus (K'_{j,t})^{\perp}$ corresponds to a locally defined, G_i -equivariant t -trivialization of the restriction of $K'_{i,t}$ to $\bar{\partial}\hat{f}_{i,i} \in V_j$ with the property that the following diagram commutes

$$\begin{array}{ccccc} K'_{j,t}(f') & \longrightarrow & K'_{i,t}(f') & \xrightarrow{A_t} & V_i/V_j \\ \uparrow & & \uparrow & \nearrow A_0 & \\ K'_{j,0}(f') & \longrightarrow & K'_{i,0}(f') & & \end{array}$$

As $\bar{\partial}\hat{f}_{i,i}$ is transverse to V_j , we may extend the above t -trivialization to a t -trivialization of $K'_{i,t}$ in a neighborhood of f . The diagram above commutes for all holomorphic curves f' . As all \mathbb{R} -nil vectors in $T_{f'}\mathcal{M}^{st} \downarrow_{\mathbf{B}_0}$ are

contained in $K_{j,t}(f')$ and the t -trivialization of $K_{j,t}$ is constant on \mathbb{R} -nil vectors, the resulting t -trivialization of $K_{i,t}$ is constant on all \mathbb{R} -nil vectors.

As $(K'_{j,t})^\perp(f')$ is contained in the kernel of $T_{f'}\Phi$, the diagram

$$\begin{array}{ccc} K_{i,t}(f') & \xrightarrow{T_{f'}\Phi} & T_{\Phi(f')}\mathbf{X} \\ \uparrow & \nearrow T_{f'}\Phi & \\ K_{i,0}(f') & & \end{array}$$

commutes for holomorphic curves f' .

So far we have constructed a Φ -submersive t -trivialization of $K_{i,t}$ on a neighborhood of f which is compatible with the t -trivialization of $K_{j,t}$. Given any $j' \prec i$ so that $\dim V_{j'} \leq \dim V_j$, the diagram

$$\begin{array}{ccccc} K_{j',t}(f') & \longrightarrow & K_{j,t}(f') & \xrightarrow{(1-t)D\pi_{V_{j'}}\bar{\partial} + t(D\pi_{V_{j'}}\bar{\partial})^c} & V_j/V_{j'} \\ \downarrow & & \downarrow & \nearrow D\pi_{V_{j'}}\bar{\partial} & \\ K_{j',0}(f') & \longrightarrow & K_{j,0}(f') & & \end{array}$$

commutes whenever $K_{j',t}(f')$ and $K_{j,t}(f')$ are both defined. It follows that our locally constructed t -trivialization of $K_{i,t}$ is automatically compatible with the t -trivialization of $K_{j',t}(f')$. Note that on the other hand, we have no reason to expect that our constructed t -trivialization is compatible with a t -trivialization of $K_{j',t}$ if $\dim V_{j'} > \dim V_j$, and we also have no reason to expect compatibility with the already defined t -trivialization on a neighborhood of \mathcal{M}' .

- (3) If f is not contained in \mathcal{M}' or $\mathcal{U}_{j,\underline{i}}$ for some $j \prec i$, then proceed as follows: On a neighborhood of f , choose an extension $K'_{i,t}$ of $K_{i,t}$. Choose a G_i -equivariant metric on $T_{\hat{f}_{i,i}}\mathcal{M}^{st}\downarrow_{\mathbf{B}_0}$ as in Remark 8.5.

For all f' in a neighborhood of f in $\hat{f}_{i,i}$, $T_{f'}\Phi : K'_{i,t}(f') \rightarrow T_{\Phi(f')}\mathbf{X}\downarrow_{\mathbf{X}_0}$ is surjective. We may therefore locally choose a G_i -equivariant splitting of $K'_{i,t}$ into

$$K'_{i,t} = (\ker T\Phi \cap K'_{i,t}) \oplus W$$

and a G_i -equivariant splitting of $T_{\hat{f}_{i,i}}\mathcal{M}^{st}\downarrow_{\mathbf{B}_0}$ into

$$T_{\hat{f}_{i,i}}\mathcal{M}^{st}\downarrow_{\mathbf{B}_0} = (\ker T\Phi \cap T_{\hat{f}_{i,i}}\mathcal{M}^{st}\downarrow_{\mathbf{B}_0}) \oplus W$$

Denote by

$$\pi : T_{\hat{f}_{i,i}}\mathcal{M}^{st}\downarrow_{\mathbf{B}_0} \longrightarrow K'_{i,t}$$

the projection which in the above splittings is the orthogonal projection of $\ker T\Phi \cap T_{\hat{f}_{i,i}}\mathcal{M}^{st}\downarrow_{\mathbf{B}_0}$ to $\ker T\Phi \cap K'_{i,t}$ and the identity on W . As our splittings and our metric are G_i -equivariant, π is a G_i -equivariant projection.

To construct a t -trivialization of $K'_{i,t}$, we shall define a connection in the t -direction of $K_{i,t}$. A section σ_t of $K'_{i,t}$ for all t may be viewed as a family of sections of $T_{\hat{f}_{i,i}}\mathcal{M}^{st}\downarrow_{\mathbf{B}_0}$. The derivative $\frac{d\sigma_t}{dt}$ is again a section of $T_{\hat{f}_{i,i}}\mathcal{M}^{st}\downarrow_{\mathbf{B}_0}$. Define

$$\nabla_t \sigma_t := \pi \left(\frac{d\sigma_t}{dt} \right)$$

Note that $\nabla_t f \sigma_t = \frac{\partial f}{\partial t} \sigma + f \nabla_t \sigma$, so ∇_t may be regarded as defining a G_i -invariant connection in the t -direction on $K'_{i,t}$. Therefore parallel transport

in the t direction gives G_i -equivariant t -trivialization maps $K'_{i,0} \rightarrow K'_{i,t}$. For holomorphic curves f' , $K_{i,t}(f')$ contains all \mathbb{R} -nil vectors, so our t -trivialization is the identity on these \mathbb{R} -nil vectors, as required. Notice too that if σ_t is a section for which $T\Phi(\sigma_t)$ is independent of t , then $T\Phi(\nabla_t \sigma)$ is the zero section. It follows that our t -trivialization commutes with $T\Phi$, so the diagram

$$\begin{array}{ccc} K_{i,t}(f') & \xrightarrow{T_f \Phi} & T\mathbf{X} \\ \downarrow & \nearrow T_f \Phi & \\ K_{i,0}(f') & & \end{array}$$

commutes.

The above three methods give us a locally defined, Φ -submersive t -trivialization of $K_{i,t}$ around every holomorphic curve in $K_{i,t}$. As these t -trivializations have no reason to match up, we shall need to average them using a G_i -invariant partition of unity. In particular, given any finite collection of t -trivializations of $K_{i,t}$ defined in a neighborhood of f , we may extend them all to locally defined t -trivializations of some extension $K'_{i,t}$ of $K_{i,t}$. These t -trivializations may then be thought of as connections on $K'_{i,t}$ in the t -direction, which may be averaged using a partition of unity to create another t -trivialization of $K'_{i,t}$. Note the following

- The resulting averaged t -trivialization of $K_{i,t}$ does not depend on the choice of extension $K'_{i,t}$ used.
- As averaging G_i -invariant connections using a G_i -invariant partition of unity gives a G_i -invariant connection, the resulting t -trivialization of $K_{i,t}$ is G_i -invariant.
- A t -trivialization is Φ -submersive if and only if for each holomorphic curve f , the corresponding connection ∇_t satisfies the following property: if σ_t is a section of $K_{i,t}(f)$ so that $T_f \Phi(\sigma_t)$ is constant, then $T\Phi(\nabla_t \sigma_t) = 0$. This property is preserved when we average connections satisfying it, so the resulting t -trivialization of $K_{i,t}$ is also Φ -submersive.
- Averaging connections which are the same on a subspace produces a connection which is the same as the original connections on the given subspace. It follows that our averaged t -trivialization is constant on \mathbb{R} -nil vectors.
- If all the original t -trivializations were compatible with a given t trivialization of $K_{j,t} \subset K_{i,t}$, then the averaged t -trivialization is also compatible with the given t trivialization. This follows because compatibility with the inclusion $K_{j,t} \subset K_{i,t}$ specifies what our connections must be restricted the subspace $K_{j,t}(f) \subset K_{i,t}(f)$, and the property of the isomorphism $K_{i,t}(f)/K_{j,t}(f) \rightarrow V_i/V_j$ being constant in our t -trivialization is also preserved by averaging.

Each holomorphic curve in the closure of $\mathcal{U}_{j,i}$ is in $\mathcal{U}_{j,\underline{i}}$. As embedded Kuranishi structures are by definition locally finite, each holomorphic curve f in $\hat{f}_{i,i}$ has an open neighborhood in $\hat{f}_{i,i}$ which intersects $\mathcal{U}_{j,i}$ for $j \prec i$ only if $f \in \mathcal{U}_{j,\underline{i}}$. Similarly, recall that we have chosen an open neighborhood O of $\mathcal{M}' \subset \mathcal{M}$ which has closure contained in the open neighborhood on which our trivialization is already defined. Our open neighborhood of f may be chosen small enough that if f is not contained in the open neighborhood for which a t -trivialization is already defined, then our open neighborhood does not intersect O . We may also choose this neighborhood of f small enough so that the relevant method above for constructing a t -trivialization applies, and choose such a t -trivialization. Choose a G_i -equivariant partition of unity subordinate to the corresponding open cover of the holomorphic curves in $\hat{f}_{i,i}$, and average our t -trivializations using this partition of unity.

As noted in the bullet points above, the corresponding t -trivialization is G_i -equivariant, Φ -submersive, agrees with the previously chosen trivialization on O , and for all $j \prec i$ is compatible with the t -trivialization of $K_{j,t}$ on $\mathcal{U}_{j,i}$. \square

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